THE
RATIONALE
OF
CIRCULATING NUMBERS,

WITH THE INVESTIGATIONS OF

All the RULES and PECULIAR PROCESSES used in that Part of DECIMAL ARITHMETIC.

TO WHICH ARE ADDED, SEVERAL CURIOUS

MATHEMATICAL QUESTIONS;

WITH SOME USEFUL REMARKS ON

Adfected EQUATIONS, and the Doctrine of FLUXIONS.

ADAPTED TO THE USE OF SCHOOLS.

By H. CLARKE.

Omnia quæcunque a primæva rerum natura constructa sunt, numerorum videntur ratione formata. Hoc enim fuit principale in animo conditoris exemplar.

BOETHIUS.

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TO

Thomas Butterworth Bayley, Esq;

OF HOPE,

FELLOW OF THE ROYAL SOCIETY.

SIR,

IT gives me the highest Satisfaction and Pleasure, that you have condescended to receive this my first Essay under your Protection. And all who are honoured with your Friendship, and are acquainted with your superior Knowledge in polite and useful Learning, in which you have justly included the Science of Numbers, will be sensible of my Happiness in being thus permitted to address you.

Were

Were my Abilities, Sir, equal to my Wiſhes, I could with Pleaſure dilate on thoſe many excellent Qualifications, adorned with the utmoſt Good-nature and Humanity, which have rendered your Character ſo conſpicuous. But, as I well know I ſhould fail in the Attempt, the only Uſe I can make of this Opportunity, is, to teſtify my Regard to ſo generous a Patron, by publicly acknowledging the many Favours which I, however undeſerving, have received at your Hands; and which I ſhall always remember with the ſincereſt Gratitude. I am,

SIR,

Your moſt obliged

And obedient Servant,

HENRY CLARKE.

PREFACE.

WHEN we conſider the Excellency and ſuperior Uſefulneſs of Decimal Arithmetic above all other Kinds of Computation, we ſhall readily allow, that an Attempt to render the more intricate Parts thereof clear and intelligible, not only merits the peculiar Attention of thoſe concerned in the Inſtruction of Youth, but is particularly intereſting to all others who deſire Accuracy in their Calculations; and is therefore ſo far from being an unneceſſary Work, that it appears to be of the greateſt Utility. But as I am ſenſible how extremely difficult it is, even in the beſt Performances of this Nature, to eſcape the Malevolence of thoſe, who fancy it their Intereſt to keep others in a long Dependence on themſelves; I ſhall beg Leave to obſerve, that I ſhall be well pleaſed, notwithſtanding their Cenſure, if my Deſign meets with the Favour of the Candid and Ingenuous, who, I am perſuaded, upon a ſufficient Peruſal, will acknowledge, that the Method here purſued is not only new, but that it is attended with all the Perſpicuity the Subject can admit of.

The principal Deſign then of this Treatiſe is, to retrench all thoſe Superfluities (as I may call them) which

Cunn

Cunn and others have loaded the Theory of Circulating Decimals with; and to shew, that the whole Business depends upon, or may be deduced from, this one single Operation, That of finding *a finite Vulgar Fraction equivalent to an infinite Repeating Decimal.* This being once understood, the Rationale of all the Rules will be obvious; and the Pupil will then go on, not only with Pleasure, but with Speed: For as the Man, who is engaged in a Race with every Obstacle removed from his Course, has undoubtedly the Advantage of him who must turn and wind to get clear of the Impediments; so not only in this, but in every Art and Science, when Difficulties are removed, as well as a concise general Rule pointed out, the Mind's chief Labour is accomplished. So Horace says,

Quicquid præcipies, esto brevis: ut citò dicta
Percipiant animi dociles, teneantque fideles.

Yet that I might not be thought to affect an unintelligible Conciseness, I have considered every Rule distinctly, illustrated them with proper Examples, and given the Investigation symbolically; by which the Scholar may at one View comprehend the whole Process.

A Question may possibly arise with some, what Advantage will result from a farther Investigation of the Nature of Decimals? Have not the most eminent Mathematicians written professedly on the Subject, and done all that was useful or curious therein? In answer to which, it will be necessary to take a retrospective View of the principal Authors who have treated upon Decimals;

mals; from whence it will clearly appear, that there is still Room for farther Improvement. The first Specimen of Decimal Arithmetic that we meet with, is in the Astronomical Tables of Arzachel, a Moor, who was very eminent in Spain about the Beginning of the eleventh Century. They are adapted to the Meridian of Toledo; and as they are calculated for the Arabian Year of the Hegira, were probably originally written in Arabic: The Persians, Moors, Arabs, and Saracens, being about that Period very famous for their Knowledge in Astronomy. In these Tables, the Places of the Heavenly Bodies are denoted by a *centesimal* Division of the great Circles of the Sphere, to which the Arabian Algorithm of Numbers was better accommodated than the Greek or Roman literal Notation which had been hitherto made Use of for the Egyptian Sexagesms in the Astronomical Tables of Ptolemy, Albategnius, Abenazra, and other ancient Writers. Gerard Vossius informs us also of a Treatise entitled *De Algorithmo*, written by Johannes de Sacro Bosco, about the Middle of the twelfth Century, who made Use of a *centesimal* Notation for the Extractions of the Square and Cube Roots. About the Year 1460, John Muller, sometimes named Regiomontanus, published his Book *De Triangulis*, in which he had constructed a Table of Sines to the Radius 10,000,000; an Account of which may be seen in the *Opus Palatinum de Triangulis*, by Otho and Rheticus. The next Improvement in this Part of Arithmetic, we find in a Treatise entitled *Arithmetica Memorativa*, composed in Latin Verse, by William Buckley, about the Year 1530, wherein he has given a Rule for extracting the Square Root of a Fraction; the Operation being nearly the same with the present Mode of extracting the

Square

Square Root of a Surd Number, excepting that it is limited to a certain Number of Cyphers: The Rule, as corrected by Dr. Wallis is,

Quadrato numero, senas præfigito Ciphras:*
Producti Quadri, Radix, per mille secetur.
Integra dat Quotiens; & pars ita recta manebit,
Radici ut vere nè pars millessima desit.

The Denominator being written under this Number, expresses the Square Root of the Fraction. Afterwards Peter Ramus, in his Arithmetic, written about the Year 1579, and published by Schoner, shews the Method of approximating to the Square and Cubic Roots of Surd Quantities, by adding Punctuations of Cyphers, exactly in the Manner we now practice. But the first Treatise written professedly on this Subject, was published at Leyden, 1585, by Simon Stevens, entitled *DISME*, or Decimals; which he tells us in his Geography, he believes to have been in Use among the Indians, and other Eastern Nations, long before the Sexagessimal Notation was introduced by Ptolemy, in the Time of M. Aurelius. After this Time, Decimals began to be frequently used in Arithmetical Calculations, and were particularly much advanced by Briggs and Gellibrand, in their *Trigonometria Britannica*; by Oughtred, in his *Clavis Mathematicæ denuò limata*; also Wingate, Baker, Kersey, and several other Authors of less Note, all contributed towards their Perfection, in their different Treatises of Arithmetic. Yet we do not find, that any Regard had been paid to the

* Referring to the Product of the Numerator and Denominator, mentioned in a former Rule.

Nature

Nature of *Infinite Circulating Decimals* before Dr. Wallis's Time. He was, in all probability, the first who distinctly considered this curious Subject, as he himself informs us in his Treatise of Infinites. But he has neither given the Demonstrations, nor shewn their Application. The latter of these Defects, Mr. Brown, in his Decimal Arithmetic, and afterwards Mr. Cunn in his Treatise of Fractions, attempted to supply, by giving Rules for their Operations. The former indeed has done this only in one single Case; but the latter has extended it to all Cases. But as these are also wanting in the main Point, namely, a Demonstration, and are moreover designedly expressed in such a Manner, as to set the Rationale of the Thing as far out of View as possible; it is necessary that either the Memory must be loaded with every Rule, or the Book be continually at Hand. Several other Authors have treated on Circulating Decimals. Martin, in his Decimal Arithmetic, has given some practical Rules, but hath not sufficiently demonstrated them. Emerson, in his Cyclomathesis, is excellent in the Theory, but has omitted the practical Part. Pardon, Vyse, Thompson, and some others, have also touched on this Subject; but as they all seem to have borrowed from Cunn, they are in the same Predicament. Malcolm and Donn are the only Authors I know of, who have treated the Doctrine of Circulates in an intelligible Manner. The principal Objection to the former is, that he is too concise, referring constantly to the Rules for Vulgar Fractions, by which that Perspicuity, which is the very Essence of a Demonstration, is lost; and the latter has omitted several Cases in Multiplication and Division, which frequently occur in Practice, and is also too cursory to afford the

* Learner

Learner a proper Idea of the Subject. All our later Books of Arithmetic pass by the Doctrine of Repetends unnoticed.

I must here beg my Readers not hastily to impute Arrogance to me, as if I rejected all that has been done on this Subject, or supposed myself capable of what so many Men of great Parts and great Learning seem to have come short of. For I acknowledge myself indebted to most of the Authors I have just mentioned, particularly to Malcolm and Emerson, who have furnished me with several useful Hints. And yet I can justly make the same Observation with the former Author, namely, That "the Rules I have given are chiefly the Effect of Speculation" made some Years ago on this Subject, before I had seen his System of Arithmetic, or even any of the before-mentioned Authors.

It is not impossible but an Objection may be raised by others, who have never adverted to the Subject proposed, as that this part of Arithmetic is of little or no no Use, since all Decimal Operations may be performed sufficiently near the Truth without it. These Persons I shall refer to the Practical Questions at the latter end of this Treatise, and only here observe, that those Instances, among innumerable others which might be produced, serve to shew, that the common Way is very defective, though carried out to seven or eight Decimal Places, and that the Method of Circulates is absolutely necessary, where any Degree of Accuracy is required. The Want therefore of a Treatise somewhat of the following Kind, which should at once shew the particular Properties of Repeating Decimals, and the Investigations of the Rules for Operation

ration from *one certain general principle*, naturally gave Occasion to the following Sheets. For as I had frequently observed in the Course of Teaching, that Youth, by their not seeing the Reason of the Thing, could not long retain the Rules in their Memory, I endeavoured to demonstrate them *vivâ voce*, and then obliged the Scholar to give the Investigation himself *in scripto*; by which I could easily discern whether he fully comprehended the Nature of them. But as the Investigation and Rule given by the Pupil were generally too prolix and incorrect to be inserted in their Cyphering-books, I drew up the following Sketch, as a general Form by which they might correct theirs, if necessary. By this Method of Procedure, I have the Pleasure to find, that a Boy, who is tolerably acquainted with the common Rules in Arithmetic, will readily acquire as clear a Knowledge of the Rules for Circulating Decimals, as he has of any common Operation in Whole Numbers.

But as the Operations of Circulates (as well as all other Arithmetical Calculations) are most easily performed by Logarithms, I have shewn the Method of finding the Logarithm of any Repeating Decimal; whereby the whole Business is greatly facilitated, and the Difficulty and Intricacy of the Rules by Common Arithmetic avoided. And, for the Amusement of such Pupils as have touched on the first Principles of Algebra and Geometry, I have inserted a few Questions, chiefly Originals, with their Solutions; and some are given without Solutions, which are intended for the Exercise of those that are farther advanced. I have also added several Remarks on those Parts

of

of the Mathematics which ~~seem to~~ the young Reader to be rather obscure, namely, On Cardan's and Colson's Theorems for Cubic Equations, wherein a very clear and concise Rule is given for extracting the Cubic Root of an impossible Binomial; by which Cardan's Theorem is rendered generally useful, in finding the Roots of an Equation when they are *all* real, as well as when there is but *one* real and two imaginary—On the improbability of obtaining general Formulæ for the Sursolid and other higher Equations—On the Method of tabulating Literal Equations, illustrated by Examples; from whence the Reversion of a Series, however affected with Radicals may be easily performed—On the direct and inverse Method of Fluxions, wherein the Principles are fully explained, and by avoiding all Metaphysical Considerations, rendered clear to the lowest Capacity. The whole Business of finding Fluxions is reduced to one general Rule; and the particular Forms of fluxionary Expressions are so distinguished, that the Learner may almost immediately determine in what Manner the Fluent may be obtained—On the Correction of a Fluent, and the Reason of it—On Trigonometrical Fluxions, with their great Importance in Astronomy—On the Phænomena of Saturn's Ring, being a new and curious Analytical Solution of the Problem respecting the Times of its appearance and disappearance; whereby is also exhibited a new Species of Curves, &c. which is extracted from a Treatise just published, entitled, *Essai sur les Phénomènes relatifs aux disparitions périodiques de l'anneau de Saturne.* By M. Dionis du Séjour, Fellow of the Royal Societies of London and Paris. In the Register of the Royal Academy of Sciences at Paris, for 1775, we have the following Encomium on

this

this Work, *Tels sont les objets que M. Du Séjour a traités dans son Ouvrage; et l'on voit qu'il n'a rien laissé à désirer sur la théorie des phases de l'anneau de Saturne. L'élégance, la finesse et la simplicité des méthodes dont il a fait usage, rendent cet Ouvrage très-intéressant pour les Géomètres; et la discussion des Phénomènes depuis* 1600 *jusqu'en* 1900, *le rend nécessaire aux Astronomes qui voudront dans la suite observer avec précision ces apparences; ainsi nous croyons qu'il mérite d'être imprimé avec l'Approbation et le Privilège de l'Académie.* Signed by M. M. d'Alembert, Le Chevalier Borda, Bézout, Vandermonde, M. de la Place, and Jean de Fouchi; some of the greatest Mathematicians now living *.—I have also added some new and useful Geometrical Propositions; and, lastly, have given a Catalogue of the most approved Authors in the several Branches of Mathematics, Philosophy, and Astronomy; from which are selected those that are generally esteemed the most useful, and ranged in the Order they may be read

* In this Treatise of Sejour's we meet with the following Paragraph; which, as it may, perhaps, convince those who still doubt the Existence of Saturn's Satellites, I have taken the Liberty to insert.—*Le quatrième Satellite fut d'abord découvert par M. Huyghens, le* 25 *Mai* 1655. *Les quatre autres Satellites ont été découverts successivement par Dominique Cassini; le cinquième sur la fin d'Octobre* 1671, *le troisième le* 23 *Décembre* 1672, *le premier et le second en* 1684. *Les Anglois contestèrent long tems l'existence de ces Satellites. Ce ne fut qu'en* 1718, *que M. Pound ayant élevé au-dessus du clocher de sa Paroisse, l'excellent objectif de cent vingt trois pieds de foyer, donné par M. Huygens à la Société royale, on fut assuré pour la première fois en Angleterre, que Saturne avoit réellement cinq Satellites. Lors de leur découverte, M. Cassini les avoit nommés* Astra Lodoicea, *par allusion à Louis XIV, à l'exemple des Satellites de Jupiter, que Galilée avoit nommés* Astra Medicea. *L'on fit frapper en France une Médaille pour consacrer cette découverte; la Médaille représentoit Saturne accompagné de ses cinq Satellites, avec l'exergue suivante;* Saturni Satellites primùm cogniti.

read by the young Student to the most Advantage. The known and received Terms in the Circulates I have retained, except in one or two Instances, where others offered themselves which seemed much more significant; as for the words *Pure Compound* and *Mixed Compound*, which evidently carry in them an Air of Absurdity, I have substituted *Pure Multiple* and *Mixed Multiple*. And for the greater Elegance and Perspicuity in the Operations, I have made use of the common accentual Dash over the Repetend; by which also that Illegibility of the Figures, often caused by the other Method, is avoided.

The Success I have met with in my own School, by using the following Rules for Circulating Decimals, and their Investigations, together with the Utility of the Mathematical Remarks, in rendering those intricate Affairs extremely easy and intelligible to the Learner, and I may with some Truth add that trite Apology, the Solicitations of several of my Friends, have induced me to publish them. And as I can manifestly have no lucrative Views by so small an Affair; so neither have I any great Anxiety about its Reception. Should its Usefulness to either Master or Scholar be a Reason for its surviving the Attacks of a carping *Zoilus*, I have acknowledged already, it would be no unpleasing Event. Should the contrary happen, I must (as Mr. Harris observes) acquiesce in its Fate; and let it peaceably pass to those destined Regions, whither our modern Productions are daily passing:

——*in vicum vendentem thus & odores.*

HOR. Lib. ii. epist. 1.

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THE

THE

RATIONALE

OF

CIRCULATING NUMBERS.

SECT. I.

The Theory of Circulates.

1. A Circulate, or recurring decimal, is that wherein one or more figures are continually repeated: they are distinguished into single and multiple, and these again into pure and mixed.

2. A pure single circulate is that which repeats a digit only; as ·666 &c. and is marked thus $\cdot\acute{6}$.

3. A pure multiple circulate is that in which several figures repeat; as ·642642 &c. marked $\cdot\acute{6}4\acute{2}$.

4. A mixed single circulate is one which consists of a terminate part, and a single repeating figure; as 4·333, &c. or $4\cdot\acute{3}$.

5. A mixed multiple circulate is that which contains a terminate part and several repeating figures; as $86\cdot\acute{3}2\acute{5}$.

6. That

6. That part of the circulate which repeats is termed the repetend; as in $2\cdot\dot{6}$, and $34\cdot\dot{6}4\dot{3}$, $\dot{6}$ and $\dot{6}4\dot{3}$ are the respective repetends.

7. In any pure circulate, the whole repeating part, being continued *ad infinitum*, is equal to a vulgar fraction, of which the numerator is the repeating number, having the decimal point removed as many places to the right-hand as there are figures in the repetend, and the denominator an equal number of *nines*. For in the pure multiple circulate $\cdot325325$, &c. *fine fine*, if we put $r =$ the repeating figures $\cdot325$, and $c =$ the whole circulating part, and from c take the $\frac{1}{1000}$ part of itself, we shall have $c - \frac{c}{1000} = \cdot325 = r$; that is,

$$\begin{array}{ll} \text{From} & \cdot325325, \text{\&c.} = c \\ \text{Take} & \cdot000325, \text{\&c.} = \frac{c}{1000} \\ \hline \text{Rem.} & \cdot325 = c - \frac{c}{1000} = r; \end{array}$$

but $c - \frac{c}{1000} = c \times \overline{1 - \frac{1}{1000}} = c \times \frac{1000 - 1}{1000} = c \times \frac{999}{1000}$; hence $c = \frac{1000 \times \cdot325}{999} = \frac{325}{999} = \cdot325325$, &c. *ad infinitum*, where there are as many *cyphers* and *nines* as repeating figures; which process evidently holds good for any pure circulating numbers.

Examples where the circulation begins in the integral part.

Ex. (1.) $\dot{3}\cdot\dot{7} = \frac{370}{99}$. (2.) $\dot{2}\cdot0\dot{6} = \frac{2060}{999}$. (3.) $\dot{4}2\cdot6\dot{3} = \frac{426300}{9999}$, &c.

Examples

Examples where the circulation begins with the prime decimal.

Ex. (1.) $\cdot\acute{3} = \frac{3}{9}$. (2.) $\cdot\acute{6}7\acute{4} = \frac{674}{999}$. (3.) $\cdot\acute{4}78\acute{6} = \frac{4786}{9999}$. (4.) $\cdot\acute{0}6\acute{4} = \frac{64}{999}$. (5.) $\cdot\acute{0}08\acute{3} = \frac{83}{9999}$, &c.

Examples where there are Cyphers, *which do not repeat, betwixt the decimal point and the first significant figure.*

Ex. (1.) $\cdot 0\acute{4} = \frac{\cdot 4}{9}$. (2.) $\cdot 00\acute{6} = \frac{\cdot 06}{9}$. (3.) $\cdot 00\acute{0}2\acute{6} = \frac{\cdot 026}{99}$. (4.) $\cdot 00\acute{7}0\acute{6} = \frac{7\cdot 06}{999}$. (5.) $\cdot 00\acute{7}14\acute{5} = \frac{71\cdot 45}{9999}$, &c.

8. Hence if the repetend (punctuated according to its places) be divided by as many *nines* as there are repeating figures, the quotient will give the whole circulating part, and the fraction is a finite expression for the series infinitely continued.

9. Hence also if any number be multiplied by an *unit* with as many *Cyphers* annexed as it contains places, and then divided by as many *nines*, it becomes a circulate; which is single or multiple according to the places in the given number, and pure or mixed as we take the whole or part of the given number for a repetend.

Pure Circulates.

Example 1.

$6 \times 10 = 60$; and $9)60.(6.66\acute{6}$, &c.

Example

Example 2.

$\dot{6} \times 10 = 6$, and $9)6(\cdot666$, &c.

Example 3.

$24 \times 100 = 2400$, and $99)2400(24'24,24$, &c.

Example 4.

$\cdot234 \times 1000 = 234$, and $999)234(\cdot234,234$, &c.

Mixed Circulates.

Example 5.

$4\cdot2\dot{6} \times 10 = 42\cdot6$, and $9)42\cdot6(4\cdot7333$, &c.

Example 6.

$327 \times 100 = 32700$, and $99)32700(330,30,30$, &c.

Example 7.

$\cdot2071 \times 1000 = 207\cdot1$, and $999)207\cdot1(\cdot2073,073$, &c.

10. Since $\frac{6}{9} = \cdot666$, &c. $\frac{234}{999} = \cdot234234$, &c. it is evident, that $\cdot666$, &c. $\times 9 = 6$; $\cdot234234$, &c. $\times 999 = 234$; from whence it appears, that if any pure circulate be multiplied by as many *nines* (considered as decimals) as it contains places, the result will be the same number complete and terminate.

11. And since $\cdot9 = 1 - \cdot1$, $\cdot99 = 1 - \cdot01$, $\cdot999 = 1 - \cdot001$, &c. it follows, that if any circulate be divided by 10, 100, &c. according to the places of the repetend, and the quotient subtracted from the given circulate, there will remain the

the ſame number, terminate and complete, that conſtituted the repetend.

Example 1.

·674,674, &c. ÷ 1000 = ·000674674, &c.

Then	from	·674674674, &c.	
	take	·000674674. &c.	
there remains		·674 - - - - -	the repetend complete.

Example 2.

3·737, &c. ÷ 100 = ·03737, &c. and 3·737, — ·03737, &c. = 3·7.

Example 3.

·00666, &c. ÷ 10 = ·00066, &c. and ·0066, &c. — ·00066, &c. = ·006.

12. A vulgar fraction, of which the denominator is any number of *nines* not leſs than the number of ſignificant figures in the numerator, is equal to a pure circulate, its repetend being the ſignificant part of the numerator. And for the punctuation obſerve, when the numerator and denominator are integral, if the places of the former exceed thoſe of the latter, the exceſs is the number of integral places in the circulate; but if the places of the latter exceed thoſe of the former, the exceſs ſhews the number of *cyphers* to be prefixed to the numerator for the repetend; and if the places of both are equal, the circulation begins with the prime decimal. If the numerator conſiſt of integers and decimals, or decimals only, there will be as many terminate *cyphers* as decimal places, prefixed to the numerator for the circulate.

Ex. (1.) $\frac{2060}{999} = \dot{2}{\cdot}0\dot{6}$. (2.) $\frac{426300}{9999} = \dot{4}2{\cdot}6\dot{3}$.

(3.) $\frac{674}{999} = {\cdot}\dot{6}7\dot{4}$. (4.) $\frac{83}{9999} = {\cdot}\dot{0}08\dot{3}$.

(5.) $\frac{.06}{9} = {\cdot}00\dot{6}$. (6.) $\frac{7{\cdot}06}{999} = {\cdot}00\dot{7}3\dot{6}$.

The truth of which appears from its being plainly the reverse of Art. 7.

13. If the numerator of a Vulgar fraction consists of complete repetends, and the denominator hath as many *nines* as the figures in the numerator, one period of the repeating figures is equivalent to the whole fraction.

Ex. (1.) $\frac{2626}{9999} = {\cdot}\dot{2}62\dot{6}$ (Art. 12.) which is evidently $= {\cdot}\dot{2}\dot{6}$.

(2.) $\frac{434{\cdot}3}{9999} = {\cdot}0\dot{4}34\dot{3} = {\cdot}0\dot{4}\dot{3}$.

14. If a vulgar fraction hath a repeating numerator, and the denominator as many *nines* as significant figures in the repetend, that fraction is equal to the sum of two or more other fractions, which may be turned into circulates by Art. 12.

Ex. (1.) $\frac{4646}{99} = \frac{4600}{99} + \frac{46}{99} = 4\dot{6}{\cdot}4\dot{6} + {\cdot}\dot{4}\dot{6}$.

(2.) $\frac{320320}{99} = \frac{320000}{99} + \frac{320}{99} = 3232{\cdot}\dot{3}\dot{2} + {\cdot}\dot{3}\dot{2}$.

(3.) $\frac{454{\cdot}5}{99} = \frac{450}{99} + \frac{4{\cdot}5}{99} = 4{\cdot}\dot{5} + {\cdot}0\dot{4}\dot{5}$.

15. A

15. A vulgar fraction, of which the numerator is *unity*, and denominator any number of *nines*, is equal to an infinite series of fractions, the numerator of the first term being an *unit*, and denominator as many *cyphers* as there are *nines* in the given fraction, with an *unit* annexed; and the succeeding terms in a geometrical progression, of which the common ratio is the first term.

Thus $\frac{1}{9} = \cdot 11111$, &c. $= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$, &c. $= \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3$, &c.

$\frac{1}{99} = \cdot 010101$, &c. $= \frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000}$, &c. $= \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3$, &c.

16. From whence it follows, that any circulating decimal may be considered as consisting of a series of fractions, of which the numerators are the repeating number, and the denominators an *unit*, with as many *cyphers* annexed as express the local value of the repetend.

Example 1.

$\cdot\dot{6} = \cdot 666$, &c. $= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000}$, &c. $= \cdot 6 + \cdot 06 + \cdot 006$, &c.

Example 2.

$\cdot\dot{4}\dot{6} = \frac{46}{100} + \frac{46}{1000}$, &c. $= \cdot 46 + \cdot 0046 + \cdot 000046$, &c.

Example 3.

$\cdot 0\dot{8}\dot{5} = \frac{85}{1000} + \frac{85}{1000000}$, &c. $= \cdot 085 + \cdot 00085 +$ $\cdot 0000085$, &c.

From whence it is also evident that any circulate, as $2\cdot\dot{3}$ will be equal $2 + \cdot\dot{3} = 2 + \cdot 3 + \cdot 03 + \cdot 003$, &c. or, $4\cdot 6\dot{2}\dot{1} = 4\cdot 6 + \cdot 0\dot{2}\dot{1} = 4\cdot 6 + \cdot 021 + \cdot 00021$, &c.

17. The sum of an infinite series of *nines* is equal to *unity* in the next left-hand place; thus, $\cdot 999$, &c. $= 1$, $\cdot 0999$, &c. $= \cdot 1$, $\cdot 00999$, &c. $= \cdot 01$, &c. For it is evident $\cdot 9 \left(= \frac{9}{10}\right)$ wants only $\frac{1}{10}$ of *unity*, $\cdot 99$ wants $\frac{1}{100}$, and $\cdot 999$, wants $\frac{1}{1000}$, and so on. So that if the series were infinitely continued, the difference between that series and *unity* would be equal to an *unit* divided by infinity; which, from the doctrine of infinites, is known to be equal *nothing*.

18. Hence it appears, that any pure circulate, being multiplied by as many *nines* as there are repeating figures, will always carry the repetend to the next left-hand place.

Example 1.

$\cdot 666 \times 9 = 5\cdot 999$, &c. $= 6. = \cdot\dot{6} \times 9 + \cdot\dot{6}$.

Example 2.

$\cdot 2424$, &c. $\times 99 = 23\cdot 999$, &c. $= 24. = \cdot\dot{2}\dot{4} \times 99 + \cdot\dot{2}\dot{4}$.

19. The

19. The circulating figures may be ſuppoſed to begin at any place in the repetend. For $2{\cdot}\dot{3}4\dot{2}342$, &c. is evidently $=$

$$2\frac{342}{999} = 2{\cdot}3\frac{423}{999} = 2{\cdot}34\frac{234}{999} = 2{\cdot}342\frac{342}{999}, \text{ \&c.}$$

20. The repeating figures of any circulating number may be conſidered as conſiſting of twice, or thrice, that number of figures; or any multiple thereof.

Thus in $5{\cdot}2434343$, &c. the repetend 43 having 2 places, may be conſidered as having 4, 6, &c. places. For ſince 43 is ſuppoſed to be repeated *for ever*, it is obvious, that $5{\cdot}2\dot{4}\dot{3} =$

$5{\cdot}2\dot{4}34\dot{3} = 5{\cdot}2\dot{4}3434\dot{3}$, &c.

21. Any number multiplied by an *unit*, with any number of *cyphers* annexed, and once ſubtracted, is manifeſtly the ſame as being multiplied by as many *nines* as there are *cyphers*. For $10a - a = 9a$, $1000b - b = 999b$.

22. A mixed circulate, ſingle or multiple, is equal to a vulgar fraction, of which the numerator is the given number multiplied by 10, 100, 1000, &c. (according to the number of repeating figures) and leſſened by its terminate part, conſidered in it's local value, and denominator as many *nines* as repeating figures.

Example 1.

$$4{\cdot}\dot{3} = \frac{39}{9}.$$

For $4{\cdot}\dot{3} = 4\frac{3}{9}$, (Art. 7.) $= \frac{4\times9+3}{9}$; but $4\times9+3 =$

$4\cdot\acute{3} \times 9$ (Art. 18.) $= 4\cdot\acute{3} \times 10 - 4\cdot\acute{3}$ (Art. 21.) $= 43\cdot\acute{3} - 4\cdot\acute{3} = 43 - 4 = 39 \therefore \frac{39}{9} = 4\cdot\acute{3}$.

Example 2.

$$5\cdot2\acute{4}\acute{3} = \frac{519\cdot1}{99}.$$

For $5\cdot2\acute{4}\acute{3} = 5\cdot2 + \cdot0\acute{4}\acute{3}$ (Art. 16.) $= 5\cdot2\ \frac{4\cdot3}{99} = \frac{5\cdot2 \times 99 + 4\cdot3}{99}$, and $5\cdot2 \times 99 + 4\cdot3 = 5\cdot2\acute{4}\acute{3} \times 99 = 5\cdot2\acute{4}\acute{3} \times 100 - 5\cdot2\acute{4}\acute{3} = 524\cdot\acute{3} - 5\cdot2 = 519\cdot1, \therefore 5\cdot2\acute{4}\acute{3} = \frac{519\cdot1}{99}$.

Or thus, $5\cdot2\acute{4}\acute{3} \times 10 = 52\cdot\acute{4}\acute{3} = \frac{5191}{99}$, (Ex. 1.); and dividing by 10, we have $5\cdot2\acute{4}\acute{3} = \frac{519\cdot1}{99}$.

Example 3.

$$\acute{2}35\cdot\acute{7} = \frac{235500}{999}.$$

For $\acute{2}35\cdot\acute{7} = 235\cdot735$ (Art. 19.) $= \frac{235\cdot735 \times 1000 - 235}{999}$ $= \frac{235\cdot7 \times 1000 - 200}{999} = \frac{235500}{999}$.

Or thus, $\acute{2}35\cdot\acute{7} \div 100 = 2\cdot\acute{3}5\acute{7} = \frac{2355}{999}$ (Ex. 1.) $=$ (by multiplying by 100) $\frac{235500}{999}$, as before.

Or

Or thus, $2\dot{3}5\cdot\dot{7} = 200 + \dot{3}5\cdot\dot{7} = 200\frac{35700}{999} = \frac{235500}{999}$; as before.

Example 4.

$$\cdot 2\dot{3}\dot{5} = \frac{23\cdot 3}{99}.$$

For $\cdot 2\dot{3}\dot{5} = \cdot 2 + \cdot 0\dot{3}\dot{5} = \cdot 2\frac{3\cdot 5}{99} = \frac{\cdot 2 \times 99 + 3\cdot 5}{99} = \frac{\cdot 2\dot{3}\dot{5} \times 100 - \cdot 2\dot{3}\dot{5}}{99} = \frac{23\cdot 5 - \cdot 2}{99} = \frac{23\cdot 3}{99}.$

Or thus, $\cdot 2\dot{3}\dot{5} \times 10 = 2.\dot{3}\dot{5} = (\text{Ex. 1.})\ \frac{233}{99}$; which being divided by 10 gives $\cdot 2\dot{3}\dot{5} = \frac{23\cdot 3}{99}$; as before.

Example 5.

$$\cdot 02\dot{7}\dot{4} = \frac{2\cdot 72}{99}.$$

For $\cdot 02\dot{7}\dot{4} = \cdot 02 + \cdot 00\dot{7}\dot{4} = \cdot 02\frac{\cdot 74}{99} = \frac{\cdot 02\dot{7}\dot{4} \times 100 - \cdot 02}{99} = \frac{2\cdot 72}{99}.$

Or, $\cdot 02\dot{7}\dot{4} \times 100 = 2\cdot\dot{7}\dot{4} = \frac{272}{99}$; this divided by 100 gives $\frac{2\cdot 72}{99}$; as before.

23. If the Denominator of a vulgar fraction, (not repeating in the numerator) consisting of *nines*, have less places than the significant figures of the numerator; that fraction is equal to a mixed circulate, of which the repetend has the same places

as

as there are *nines* in the denominator; and its circulate value may be thus found: Expunge the denominator, and remove the decimal point as many places to the left-hand as there are *nines* in the denominator; then dash off the same number of significant figures for a repetend; and the remaining figures to the left will be the terminate part. Add the terminate part to the numerator of the given fraction, observing to encrease the right-hand place by 1, if there be 1 carried in the addition, the sum being dashed for a repetend, and the decimal point removed as before, will give the correct circulate equal to the given fraction. The reason of which is evident from the last article.

Example 1.

Let $\frac{519{\cdot}1}{99}$ be proposed.

Which being ordered according to the rule will stand thus;

$$\frac{519{\cdot}1}{99},\ 519{\cdot}1,\ 5{\cdot}191,\ 5{\cdot}1\dot{9}\dot{1},\ 5{\cdot}1\dot{9}\dot{1},\ +\ 5{\cdot}2 = 524{\cdot}3\ \therefore\ 5{\cdot}2\dot{4}\dot{3} = \frac{519{\cdot}1}{99}.$$

Example 2.

$$\frac{235500}{999},\ 235500,\ 235{\cdot}5,\ 2\dot{3}5{\cdot}\dot{5},\ 235500 + 200 = 235700\ \therefore\ 2\dot{3}5{\cdot}\dot{7} = \frac{235500}{999}.$$

Example 3.

$$\frac{2{\cdot}72}{99},\ 2{\cdot}72,\ {\cdot}0272,\ {\cdot}02\dot{7}\dot{2},\ 2{\cdot}72 + {\cdot}02 = 2{\cdot}74\ \therefore\ {\cdot}02\dot{7}\dot{4} = \frac{2{\cdot}72}{99}.$$

24. Any

24. Any circulating number being multiplied by a given number, the product will be a circulating number, of which the repetend will have the same places as before. For every repetend being equally multiplied must produce equal products.

Example 1.

$\cdot\dot{3} \times 2 = \cdot 6$ hence

$$\begin{array}{l} \quad\;\; 6 \\ \quad\; 6 \\ \;\; 6 \\ \hline \cdot 666, \text{ \&c.} = \cdot\dot{6} \end{array}$$

But, if the product consist of more places than the given number, the overplus belongs to the first place of the next repetend.

Example 2.

$\cdot\dot{6} \times 4 = 2\cdot 4$, hence

$$\begin{array}{l} \quad\;\; 24 \\ \quad 24 \\ 24\;\; 2 \\ \hline 2\cdot 666 \text{ \&c.} = 2\cdot\dot{6}. \end{array}$$

Example 3.

$\cdot\dot{2}3\dot{4} \times 13 = 3\cdot 042$, therefore

$$\begin{array}{l} \qquad\quad 3042 \\ \quad\;\; 3042 \\ 3\;042 \qquad 3 \\ \hline 3\cdot 045045045 \text{ \&c.} = 3\cdot\dot{0}4\dot{5}. \end{array}$$

25. Any vulgar fraction, of which the numerator and denominator are prime to each other (neither 2 nor 5) being reduced to a decimal, will be a circulate, and the places of the repetend

repetend always less than the number of *units* in the denominator. For it is evident, the remainder is always less than the divisor, one of which must therefore return a second time, when the quotient consists of as many places as there are *units* in the divisor, if not before.

Example 1.

Let the vulgar fraction $\frac{3}{7}$ be proposed.

```
7)3·0(428571,428571,4, &c.
  28
  --
   20
   14
   --
    60
    56
    --
     40
     35
     --
      50
      49
      --
       10
        7
       --
        3 &c.
```

Here all the remainders are 1, 2, 3, 4, 5, 6, one of which (3) returns in the 7th place; therefore the repetend consists of 6 places.

Example 2.

Let $\frac{9}{11}$ be proposed.

```
11)9·0(81,81, &c.
   88
   --
    20
    11
    --
     9 &c.
```

Here the remainders are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; and as one of them returns in the 3d place, the repetend has but 2 places.

26. Any

26. Any vulgar fraction, of which the denominator is a prime, (except 2 and 5) being thrown into a decimal, will have as many circulating figures as the denominator would require *nines* for a dividend till it terminates. For if we divide 1000 &c. by any number till 1 remains, the quotient will evidently begin to repeat; but 999 &c. is only 1 less than 1000 &c. therefore *nothing* must remain when the figures have once circulated.

It is evident from the last article, that, whatever the dividend be, provided it be prime to the divisor, the number of circulating figures will be the same.

Example 1.

Let $\frac{3}{11}$ be proposed.

$$\begin{array}{l} 11)99(9 \\ \underline{99} \\ 0 \end{array}$$

Here are 2 nines used in the dividend, therefore the repetend has 2 places.

Proof

$$\begin{array}{l} 11)3{\cdot}0({\cdot}2727 \text{ \&c.} = {\cdot}\dot{2}\dot{7}. \\ \underline{22} \\ \ 80 \\ \ \underline{77} \\ \ \ 3 \text{ \&c.} \end{array}$$

Example 2.

Suppose $\frac{6}{7}$ be given.

7)999999(142857. Here are 6 nines required: hence the repetend consists of 6 places.

7)6·0

$$7)6{\cdot}0(\cdot857142,857 \text{ \&c.} = \cdot\dot{8}5714\dot{2}.$$

Proof

```
7)6·0(·857142,857 &c.
  56
  --
   40
   35
   --
    50
    49
    --
     10
      7
     --
      30
      28
      --
       20
       14
       --
        6 &c.
       --
```

Example 3.

Let $\frac{483}{37}$ be proposed.

37)999(27. Here are 3 nines made use of; therefore, the repetend has 3 places.

Proof

37)483·0 &c.(13·054,054, &c. $=$ 13·$\dot{0}5\dot{4}$.

27. If two numbers be prime to each other, the least common dividend, or number which exactly contains them, is expressed by their product; but if they be not primes, express them fractionally, and reduce the fraction to its lowest terms, the reciprocal product of the numerator and denominator of the two fractions is the least common dividend.

Example

Example 1.

Let the numbers proposed be 9 and 7.

9 and 7 are prime to each other; therefore, $9 \times 7 = 63$ is the least number which contains 9 and 7 without a remainder.

Example 2.

Let 8 and 12 be proposed.

These numbers are not primes; therefore, $\frac{8}{12} = \frac{2}{3}$ (in it's lowest terms) and 2×12, or $8 \times 3 = 24$, the least common dividend.

28. If there be three or more numbers, find the least number divisible by any two of them, then the least number divisible by this and any other, and so on; the last number found will be the least common dividend.

Example.

What is the least common dividend of 2, 3, 4, and 8?

2 is prime to 3, $\therefore 2 \times 3 = 6$; 6 is not prime to 4, $\therefore \frac{6}{4} = \frac{3}{2}$, and $2 \times 6 = 12$; 12 is not prime to 8, $\therefore \frac{12}{8} = \frac{3}{2}$, and $2 \times 12 = 24$, the least common dividend. The reason of this is evident from Euc. 37, 38. 7.

29. If the denominator of a fraction be compounded of two or more different primes (neither 2 nor 5) the least common dividend of all the numbers, for each single prime (found by the two last Art.) will express the number of circulating figures.

Thus

Thus let the proposed fraction be $\frac{3}{77} = \frac{3}{7 \times 11}$; then, since $\frac{3}{7}$ $= \cdot\dot{4}2857\dot{1}$ repeats in 6 places, and $\frac{3}{11} = \cdot\dot{2}\dot{7}$ repeats in 2 places, and the least common dividend of 6 and 2 is 6; therefore, we may suppose them both to circulate in 6 places, (Art. 20.) But 99 is divisible by 11; therefore 99,99,99, must also be divisible by 11: and since 999999 is also divisible by 7, it must therefore be divisible by $7 \times 11 = 77$; consequently, the repetend will consist of 6 places.

Example.

77)99999(12987,
77
—
229
154
—
759
693
—
669
616
—
539
539
—
...

Where there are 6 nines used before it terminates.

Hence the vulgar fraction $\frac{3}{2849}$ will have a repetend of 6 places. For $\frac{3}{2849} = \frac{3}{11 \times 7 \times 37}$; the repetend by 11, 7, and 37, hath 2, 6, and 3 places respectively; and the least number divisible by these is 6; which is the number of places in the repetend.

Example.

Example.

```
2849)999999(351
      8547
      -----
      14529        Where it terminates with 6
      14245        nines.
      -----
        2849
        2849
        ----
        ....
```

30. And generally, if the denominator of a vulgar fraction $\frac{N}{D}$ be compounded of the prime numbers A B C &c. which circulate in *a b c* &c. places respectively; and P be the least number which *a b c* &c. measure; then will $\frac{N}{A \times B \times C\,\&c.} = \frac{N}{D}$ thrown into a decimal, repeat in P places, which is obvious from the last Article.

31. If any circulating numbers, consisting of pure or mixed repetends, of equal places, be added together, the sum will have a repetend of the same number of places. For every column of repetends must evidently give the same sum.

Example 1.	Example 2.
13·054,054, &c.	2·7134,7134, &c.
2·342,342, &c.	·5428,5428, &c.
	71·4371,4371, &c.
Sum, 15·396,396, &c.	Sum, 74·6934,6934, &c.

 32. If

32. If any multiple repetends contain a certain number of places, and the places of the other repetends are found in a Geometrical progreſſion, of any ratio, the ſum will have a repetend conſiſting of as many places as the greateſt term of the progreſſion. For ſuppoſe the greateſt repetend to circulate in 8 places, the next leſs repetend can only circulate in 4, and the next in 2 places, to be conterminous (Art. 20.); and 2, 4, 8 are in Geometrical proportion.

Example 1.

8·72,72,72,72,72 &c.	Here 8, 4, 2 are the places of the repetends; each of which may therefore be ſuppoſed to conſiſt of 8 places.
74·12 74,12 74,12 &c.	
2·19 36 42 57,19 &c.	
85·04 83 28 04,04 &c.	

Example 2.

·2342342342 34,234 &c.	Here the repetends conſiſt of 3, 6 and 12 places; which being in Geometrical progreſſion, may be conſidered as conſiſting of 12 places each; therefore the ſum conſiſts of 12 places.
·410725410725,410 &c.	
·983210645271,983 &c.	
1·628170290231,628 &c.	

33. If the repetends to be added have unequal places, ſuppoſe them all to begin together, (Art. 19.) then will the leaſt common dividend of thoſe numbers give the places of repeating figures in the ſum. For it is evident that each period of repetends muſt then terminate together. (Art. 20.)

Example

Example 1.

72·32 52·643 1·04 4·714	These repetends have 2, 3, 4 places, and the least number divisible by these is 12; therefore each repetend may be supposed to consist of 12 places; which evidently cannot begin together till all the numbers become circulates.

Example 2.

78·141414 &c. 3·817817 &c. 21·737373 &c.	Here the repetends have 2 and 3 places, and the least number divisible by these is 6, which therefore denotes the places of the repetend in the sum.

34. The difference of any two circulating numbers, of which the repetends can be made conterminous, will also be a circulating number of the same places as the greater repetend. This plainly follows from the last Article.

Example 1.

From	82·8546,8546, &c.	The greater repetend consists of 4 places; so does the remainder.
Take	8·72,72,72,72, &c.	
Rem.	71·1274,1274, &c.	

Example 2.

From	7·382,382,382,382, &c.	Where the remainder has 6 places, the same as the greater repetend.
Take	4·125 368,125 368, &c.	
Rem.	3·257 014,257 014, &c.	

Examples for the Learner's Exercise.

Required the finite values of the following infinite repeating decimals.

Ex. (1.) $\cdot\acute{3}$. (2.) $\cdot\acute{4}\acute{5}$. (3.) $\cdot\acute{7}4\acute{1}$. (4.) $\cdot\acute{5}68\acute{6}$. (5.) $\cdot\acute{0}\acute{7}$. (6.) $\cdot\acute{0}04\acute{6}$. (7.) $\cdot\acute{0}00\acute{3}$. (8.) $\acute{2}\cdot\acute{6}$. (9.) $\acute{4}\cdot0\acute{7}$. (10.) $\acute{7}62\cdot\acute{3}$. (11.) $\acute{1}00\cdot0\acute{6}$. (12.) $\cdot0\acute{7}$. (13.) $\cdot00\acute{3}$. (14.) $\cdot003\acute{0}\acute{7}$. (15.) $\cdot00\acute{6}3\acute{2}$. (16.) $\cdot0\acute{7}000\acute{4}$. (17.) $2\cdot\acute{6}$. (18.) $2\cdot3\acute{4}\acute{7}$. (19.) $7\acute{2}3\cdot\acute{6}$. (20.) $\cdot\acute{6}\acute{1}3$. (21.) $\cdot03\acute{6}\acute{4}$. (22.) $\cdot005\acute{6}3\acute{2}$. (23.) $1212\cdot\acute{1}\acute{2} + \acute{1}\cdot\acute{2}$. (24.) $\acute{7}\cdot\acute{6} + \cdot0\acute{7}\acute{6}$.

Required the law of the infinite series, or circulate decimal, equivalent to each of the following vulgar fractions.

$\frac{3}{9}$; $\frac{45}{99}$; $\frac{741}{999}$; $\frac{5686}{9999}$; $\frac{7}{99}$; $\frac{46}{9999}$; $\frac{3}{9999}$; $\frac{260}{99}$; $\frac{4070}{999}$; $\frac{7623000}{9999}$; $\frac{10006000}{99999}$; $\frac{\cdot7}{9}$; $\frac{03}{9}$; $\frac{3\cdot07}{9\,99}$; $\frac{6\cdot32}{9\,99}$; $\frac{7000\cdot4}{9999\,9}$; $\frac{24}{9}$; $\frac{232\cdot4}{99}$; $\frac{722900}{999}$; $\frac{60\cdot7}{99}$; $\frac{3\cdot61}{99}$; $\frac{5\cdot627}{999}$; $\frac{120120}{99}$; $\frac{767\cdot6}{99}$;

SECTION

SECTION II.

Addition of Circulates.

CASE I.

When the numbers to be added are pure, or mixed single *circulates.*

RULE.

MAKE them all end together; (Art. 33.) then add as in common numbers, only increase the right-hand place of decimals by as many *units* as there are *nines* in that column, and the last figure will be the repetend. Or, continue the repetends two or three places forward, and add as before, observing to reject the superfluous Figures to the right-hand of the repetend.

Example.

$24{\cdot}6\dot{8}$		$24{\cdot}688\dot{8}$	88
$7{\cdot}23\dot{4}$		$7{\cdot}234\dot{4}$	44
$1{\cdot}6$		$1{\cdot}600\dot{0}$	00
$\cdot7\dot{3}$	Reduced	$\cdot733\dot{3}$	33
$28{\cdot}5\dot{6}$		$28{\cdot}566\dot{6}$	66
$4{\cdot}\dot{7}40\dot{1}$		$4{\cdot}740\dot{1}$	11
$19{\cdot}\dot{6}$		$19{\cdot}666\dot{6}$	66
	Sum	$87{\cdot}230\dot{1}$.	

For

For $19.\dot{6} = 19.666\dot{6}$; $28.5\dot{6} = 28.566\dot{6}$, &c. (Art. 4.) and $\dot{6} + \dot{1} + \dot{6} + \dot{3} + \dot{0} + \dot{4} + \dot{8} = \frac{6}{9} + \frac{1}{9} + \frac{6}{9} + \frac{3}{9} + \frac{4}{9} + \frac{8}{9}$ (Art. 7.) $= \frac{28}{9} = 3\frac{1}{9} = 3 + \dot{1}$.

Examples for the Learner's Exercise.

(1.)	(2.)	(3.)	(4.)
$21.4\dot{3}$	$716.\dot{4}$	$1468.$	$6.\dot{6}$
$67.\dot{8}$	$271.\dot{6}$	$21.\dot{4}$	$.\dot{9}$
$2.1\dot{4}$	$.\dot{8}$	$871.2\dot{3}$	$.\dot{7}$
$66.$	$4.6\dot{2}$	$.00\dot{1}$	$.\dot{2}$
$4.\dot{1}$	$4\dot{6}.$	$41.\dot{2}$	$\dot{4}.$
$34.8\dot{1}$	$7.1\dot{2}$	$6.1\dot{8}$	$.0\dot{6}$
$68.\dot{7}$	$.08\dot{2}$	$129.61\dot{4}$	$.00\dot{5}$

CASE II.

If the numbers be pure, or mixed multiple *circulates.*

RULE.

Make them all end together, (Art. 33.) then add as in common numbers, only to the right-hand figure add as many units as are carried from the column where the circulation begins, and mark off the repetend from the same place. Or, continue each repetend two or three places forward as before, rejecting the figures to the right-hand of the place where they become conterminous.

Example,

Example.

4·875		4·875875875875	75
16·32		16·323232323232	23
428·76		428·766666666666	66
31·216	Reduced	38·216216216216	16
14·6184		14·618461846184	18
·041		·041414141414	41
·4142		·414241424142	14
	Sum	503·256108593732,	

The places of repetends are 1, 2, 3, 4, and the leaſt number diviſible by theſe is 12. (Art. 28.) Each repetend may therefore be ſuppoſed to conſiſt of 12 Places (Art. 20.) and the circulation muſt evidently begin from the column where they all begin to repeat together,

The reaſon of adding the units to the right-hand figure of the ſum, is evident from continuing the repetends a few places forward.

Examples

Examples for the Learner's Practice.

(1.)	(2.)	(3.)	(4.)
14·12	4·127	127·14	·0011
4·5	16·2	6·	1·71
·682	·412	4·121	·14
·4	4·12	28·21	6·214
3·21	·016	6·002	·182
18·4	17·41	5·16	4·12

A General Rule.

Having made the repetends conterminous, divide their sum by as many *nines* as they contain places, and carry the quotient to the next left-hand column to where the circulation begins; then add as in common numbers, and put down the remainder for the repetend, observing to prefix cyphers (if necessary) to make it equal in places to the other repetends.

Example.

67·51		67·515
4·1		4·111
23·401	Reduced	23·401
4·62		4·622
17·02		17·022
3·83		3·833
	Sum	120·505

Here

Here the places of the repetends are 1, 2, and the least number divisible by these is 2, hence each repetend will have two places to become conterminous; and the sum of the repetends is plainly $\frac{15}{99}+\frac{11}{99}+\frac{1}{99}+\frac{22}{99}+\frac{22}{99}+\frac{33}{99}=\frac{104}{99}=1\,\frac{5}{99}$ $=1+\dot{0}\dot{5}$, (Art. 12.)

Examples for the Learner's Exercise.

(5.)	(6.)	(7.)	(8.)
4·6178	12·01	46·1	44·4
17·4	4·151	2·1	2·8
5·162	18·2	27·	4·127
·0171	·008	·08	·0098
2·45	1·21	4·164	18·1
16·817	42·684	21·67	1·68

(9.)	(10.)	(11.)	(12.)
·734	24·24	·01	41·
·48	1·8	41·	2·4
·7	17·81	·682	18·81
·681	5·481	18·4	·014
·2410	76·281	·6871	·004
·0007	7·41	21·48	28·65

SECTION

SECTION III.

Subtraction of Circulates.

CASE I.

If the numbers consist of pure, or mixed single *circulates.*

RULE.

MAKE them conterminous, and subtract as in common numbers; the right-hand figure will be the repetend. But if the repetend of the subducend be greater than that of the minuend, instead of 10 add only 9 to the less repetend and subtract as before. Or, continue the repetends two or three places forward, and subtract as in common numbers, rejecting the figures to the right-hand of the repetend.

Ex. 1.		Ex. 2.			
From	$24{\cdot}6\dot{7}$	$71{\cdot}4\dot{2}$	Reduced	$71{\cdot}4\dot{2}$	2
Take	$18{\cdot}4\dot{3}$	$38{\cdot}\dot{7}$		$38{\cdot}7\dot{7}$	7
Rem.	$6{\cdot}2\dot{4}$			$32{\cdot}6\dot{4}$	

For (Ex. 2.) $71{\cdot}4\dot{2} - 38{\cdot}7\dot{7} = 71{\cdot}4\frac{2}{9} - 38{\cdot}7\frac{7}{9} = 32{\cdot}6\frac{4}{9}$ $= 32{\cdot}6\dot{4}$. (Art. 7.).

Examples

Examples for the Learner's Practice.

	(1.)	(2.)	(3.)	(4.)
From	$68{\cdot}43\dot{2}$	$7{\cdot}4\dot{1}$	$54{\cdot}2\dot{1}$	$271{\cdot}\dot{4}$
Take	$42{\cdot}2\dot{7}$	$2{\cdot}\dot{8}$	$31{\cdot}6\dot{2}$	$39{\cdot}8\dot{7}$

CASE II.

When the numbers are pure, or mixed multiple *circulate.*

RULE.

Make them both conterminous and ſubtract as in common numbers; the figures in the remainder, which ſtand under the given repetends, will be the repetend of the difference. If the repetend of the ſubducend be greater than that of the minuend, leſſen the laſt figure in the remainder by one, and it will be the true repetend. Or, continue the repetends a few places forward, and ſubtract as before.

Example 1.

From	$27{\cdot}\dot{2}8\dot{3}$	Reduced	$27{\cdot}2\dot{8}3283\dot{2}$	83
Take	$13{\cdot}1\dot{2}\dot{2}$		$13{\cdot}1\dot{2}3232\dot{3}$	23
		Remains	$14{\cdot}1\dot{6}0050\dot{9}$.	

Example 2.

From	127·46	Reduced	127·4627462746	27
Take	48·6		48·6486486486	48
		Remains	78·8140976259.	

In Ex. 2. the places of the repetends are 3, 4; and the least number divisible by these is 12; which is therefore the number of repeating figures in the remainder.

The reason of taking *one* from the last figure of the remainder is evident from continuing the figures a few places forward.

Examples for the Learner's Practice.

	(3.)	(4.)	(5.)	(6.)
From	24·17	32·18	178·1	271·64
Take	12·4	8·41	24·6	41·6

	(7.)	(8.)	(9.)	(10.)
From	1·46	17·	8·41	·281
Take	·08	·412	2·418	18.

SECTION

SECTION IV.

Multiplication of Circulates.

CASE I.

When one factor is a terminate, and the other a pure circulate.

RULE.

MULTIPLY the repetend and terminate together; then remove the decimal point as many places to the right-hand as there are repeating figures, and divide the product by the ſame number of *nines*, continuing the diviſion till it repeats or terminates, which will then be the true product. Or, inſtead of dividing by the *nines*, add the product to itſelf, removing it as many places forward as exceed the number of places in the repetend by one; and thus proceed till the reſult laſt added be carried beyond the firſt: laſtly, add theſe ſeveral products together, beginning under the right-hand place of the firſt, the ſum will be the true product, having a repetend equal in places to the given circulate.

1. A ſingle terminate, and a pure ſingle circulate.

Example.

Multiply $\cdot\dot{8}$ by 7.

$$\begin{array}{r} \cdot\dot{8} \\ 7 \\ \hline 9)56(6\cdot22, \text{\&c.} = 6\cdot\dot{2} \end{array}$$

Or thus,

$$\begin{array}{r} \cdot\dot{8} \\ 7 \\ \hline 56 \\ 56 \\ 5 \\ \hline 6\cdot\dot{2} \end{array}$$

For

For $\cdot\dot{8} \times 7 = \frac{8}{9} \times 7$ (Art. 7.) $= \frac{56}{9} = 56 \times \frac{1}{9} =$ $56 \times : \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$, &c. $= 5\cdot6 + \cdot56 + \cdot056$, &c.

Examples for the Learner's Practice.

Multiply $\cdot\dot{6}$ by 2. $\cdot\dot{5}$ by 4; $\cdot\dot{7}$ by 3; $\cdot\dot{8}$ by 5; $\cdot0\dot{7}$ by $\cdot04$; $\cdot00\dot{3}$ by 60; $\cdot00\dot{5}$ by 400; $\cdot000\dot{6}$ by 20.

2. A terminate consisting of several figures, and a pure single circulate.

Example.

Multiply 23.4 by $\cdot\dot{7}$.

$$\begin{array}{l} 23\cdot4 \\ \;\;\underline{\cdot\dot{7}} \\ 99)163\cdot8(18\cdot\dot{2} \end{array} \qquad \text{Or thus,} \quad \begin{array}{r} 16\cdot3\;8 \\ 1\;6\;38 \\ 1\;63 \\ 16 \\ \underline{1} \\ 18\cdot1\;\dot{9} \end{array} = 18\cdot\dot{2}, \text{ (Art. 17.)}$$

For $\cdot\dot{7} = \frac{7}{9} = 7 \times \frac{1}{9} = 7 \times : \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$, &c. (Art. 15.)

$\therefore \cdot\dot{7} \times 23\cdot4 = 163\cdot8 \times : \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$, &c. $= 16\cdot38 +$ $1\cdot63 + \cdot1638$, &c.

For the Learner's Practice.

Multiply 21 by $\cdot\dot{3}$; 643 by $\cdot\dot{7}$; $\cdot265$ by $\cdot0\dot{6}$; $\cdot0281$ by $\dot{2}\cdot\dot{2}$; $4\cdot001$ by $\cdot00\dot{4}$.

3. A single terminate, and a pure multiple circulate.

Example.

Multiply $\acute{3}\cdot4\acute{1}$ by $\cdot4$

$$\begin{array}{r} 3\cdot41 \\ \cdot4 \\ \hline 999)1364(1\cdot365,365,\ \&c. \end{array}$$

Or thus,

$$\begin{array}{r} 1\cdot364 \\ 13 \\ 1 \\ \hline 1\cdot\acute{3}6\acute{5} \end{array}$$

For $\acute{3}\cdot4\acute{1} \times \cdot4 = \frac{3410}{999} \times \cdot4 = \frac{1364}{999} = 1364 \times \frac{1}{999} = 1364 \times : \frac{1}{1000} + \frac{1}{1000000}$, &c. $= 1\cdot364 + \cdot001364$, &c.

For the Learner's Exercise.

Multiply $\acute{2}\cdot\acute{1}$ by 8; $\cdot\acute{7}1\acute{4}$ by $\cdot6$; $\acute{7}2\cdot\acute{1}$ by $\cdot02$; $\acute{4}6\cdot\acute{8}$ by $\cdot004$. $\cdot\acute{0}04\acute{8}$ by $\cdot4$; $\cdot\acute{1}\acute{8}$ by $\cdot01$.

4. A terminate of several figures, and a pure multiple circulate.

Example.

Multiply $48\cdot76$ by $\acute{3}2\cdot4\acute{1}$.

$$\begin{array}{r} 48\cdot76 \\ 32\cdot41 \\ \hline 9999)15803116(1580\cdot4696,46,\ \&c. \end{array}$$

Or thus,

$$\begin{array}{r} 1580\cdot3116 \\ 15803 \\ 1 \\ \hline 1580\cdot\acute{4}69\acute{6} \end{array}$$

For

For $\dot{3}2{\cdot}4\dot{1} \times 48{\cdot}76 = \frac{324109}{9999} \times 48{\cdot}76 = \frac{15803116}{9999} =$ $1580{\cdot}3116 + {\cdot}15803116 +$ &c.

For the Learner's Exercise.

Multiply $\dot{2}7\dot{6}$. by $\dot{2}{\cdot}\dot{1}$; $4{\cdot}68$ by ${\cdot}\dot{0}\dot{4}$; ${\cdot}071$ by $\dot{4}{\cdot}8\dot{4}$; $\dot{2}8{\cdot}\dot{4}$ by 21; ${\cdot}\dot{0}0\dot{4}$ by $\dot{1}0\dot{8}$; $\dot{4}{\cdot}\dot{8}$ by ${\cdot}006$.

CASE II.

If one factor be a terminate, and the other a mixed circulate.

RULE.

Multiply the circulate by an *unit*, with as many *cyphers* annexed as the repetend has places, from which subtract the terminate part; then multiply the remainder by the given terminate, and divide the product by as many *nines* as the circulate has repeating figures; or, add the products as directed in Case I.

1. A single terminate and a mixed single circulate.

Example.

Multiply $4{\cdot}2\dot{3}$ by 6

	Or, thus,	
$4{\cdot}2\dot{3}$		$4{\cdot}2\dot{3}$
10		6
$42{\cdot}\dot{3}$		$25{\cdot}38$
$4{\cdot}2$		2
$38{\cdot}1$		$25{\cdot}40$.
6		
9)228·6		
$25{\cdot}4$		

For

For $4{\cdot}2\acute{3}\times 6 = \dfrac{\overline{4{\cdot}23\times 10 - 4{\cdot}2}\times 6}{9}$ (Art. 22.) $= \dfrac{228{\cdot}6}{9}$
$= 25{\cdot}4.$

And $4{\cdot}2\acute{3}\times 6 = \overline{4{\cdot}2 + \dfrac{{\cdot}3}{9}} \times 6$ (Art. 16.) $= 25{\cdot}2 + {\cdot}2.$

For the Learner's Exercise.

Multiply $4{\cdot}\acute{2}$ by 7; $27{\cdot}\acute{6}$ by ${\cdot}3$; ${\cdot}10\acute{8}$ by ${\cdot}04$; $14{\cdot}\acute{8}$ by ${\cdot}1$; ${\cdot}27\acute{1}$ by ${\cdot}005$; ${\cdot}9$ by ${\cdot}31\acute{6}$.

2. A single terminate, and a mixed multiple circulate.

Example.

Multiply $4{\cdot}2\acute{6}\acute{4}$ by ${\cdot}3$.

$$\begin{array}{r} 4{\cdot}264 \\ 100 \\ \hline 426{\cdot}4 \\ 4{\cdot}2 \\ \hline 422{\cdot}2 \\ 3 \\ \hline 1{\cdot}2666 \\ 1266 \\ 12 \\ \hline 1{\cdot}27\acute{9}\acute{3}. \end{array}$$

Or thus,

$$\begin{array}{r} 4{\cdot}264{,}64, \&c. \\ {\cdot}3 \\ \hline 1{\cdot}2793{,}93, \&c. \end{array}$$

For $4{\cdot}2\acute{6}\acute{4} \times {\cdot}3 = \dfrac{\overline{4{\cdot}264\times 100 - 4{\cdot}2}\times{\cdot}3}{99} = \dfrac{126{\cdot}66}{99}$
$= 1{\cdot}2666 + {\cdot}012666 + {\cdot}000126$, &c.

The other method is obvious.

For the Learner's Exercise.

Multiply $2\cdot\acute{4}1\acute{6}$ by 4; $\cdot8\acute{7}\acute{1}$ by $\cdot3$; $\cdot01\acute{4}\acute{6}$ by $\cdot7$; 2 by $\cdot\acute{4}\acute{7}$; $\cdot008$ by $3\cdot\acute{8}7\acute{1}$.

3. A terminate consisting of several figures, and a mixed single circulate.

Example.

Multiply $2\cdot1\acute{6}$ by $4\cdot65$

<table>
<tr><td>2·16
10
——
21·6
2·1
——
19·5
4·65
——
9)90·675
——
10·075</td><td>Or thus,</td><td>4·65×2·1
·6
——
9)2·790
——
·310
465
9 30
——
10·075</td><td>Or thus,</td><td>2·166, &c.
4·6́5
——
1083́
130́00
866́66
——
10·0749́.</td></tr>
</table>

For $2\cdot1\acute{6} \times 4\cdot65 = \dfrac{\overline{2\cdot16 \times 10 - 2\cdot1} \times 4\cdot65}{9} = \dfrac{90\cdot675}{9}$ $= 10\cdot075$.

And $2\cdot1\acute{6} = 2\cdot1 + \cdot0\acute{6}$ (Art. 16.) $= 2\cdot1 + \dfrac{\cdot6}{9}$, $\therefore$ $2\cdot1 \times 4\cdot65 + \dfrac{\cdot6}{9} \times 4\cdot65 = 10\cdot075$.

The other process is evident.

For

For the Learner's Practice.

Multiply $48\cdot\dot{7}$ by 874; $2\cdot8\dot{3}$ by ·21; $48\cdot\dot{2}$ by ·084; 48·76 by $21\cdot\dot{8}$; ·016 by $\cdot00\dot{8}\dot{7}$.

4. A terminate of several figures, and a mixed multiple circulate.

Example.

Multiply ·467 by $3\dot{2}\cdot\dot{4}$.

	Or thus,	
32·4		32·4,2, &c.
100		·467
3240		$22\dot{6}\dot{9}$
3		$194\dot{5}\dot{4}$
3210		$12\dot{9}\dot{6}96$
467		$15\cdot14\dot{2}\dot{1}$
14·9907		
1499		
14		
$15\cdot14\dot{2}\dot{1}$		

For $3\dot{2}\cdot\dot{4} \times \cdot467 = \dfrac{\overline{32\cdot4 \times 100 - 30} \times \cdot467}{99} = 14\cdot9907$ $+ \cdot149907 + \cdot00149$, &c.

The other method is plainly derived from Art. 19. 24.

For the Learner's Exercise.

Multiply 271·4 by $4\cdot8\dot{7}\dot{1}$; 271·6 by $1\cdot\dot{4}\dot{8}$; ·487 by $\cdot2\dot{7}\dot{6}$; 27·14 by $4\cdot8\dot{4}1\dot{6}$; $\cdot00\dot{1}\dot{9}$ by 2·14; $\cdot7\dot{1}\dot{4}$ by 4·876.

CASE III.

The factors, a pure circulate, and a mixed circulate.

RULE.

Multiply the mixed circulate by an *unit* with as many *cyphers* annexed as the repetend has places, and from the product subtract the terminate part; then multiply the remainder by the other repetend, having its decimal point removed as far to the right as it contains places, and reserve the result. Write down as many *nines* as there are figures in the greater repetend, and annex as many *cyphers* as the places in the less repetend, from which subtract as many *nines*, and divide the reserved result by the remainder; the quotient, continued till it terminates or repeats, is the true product. Or, instead of finding a divisor, place the result under itself according to the places of either repetend as in Case I; and then place the sum again under itself as before, observing the places of the other repetend; or divide it by as many *nines*.

1. A pure single circulate, and a mixed single circulate.

Example.

Multiply $2 \cdot \dot{3}$ by $\cdot \dot{7}$.

$2 \cdot 3$
10
———
23
2
———
$90 \quad 21$
$9 \quad\; 7$
———
$81\,)\,147\,(\,1 \cdot 814,8$, &c.

Or thus,

$14 \cdot 7$
$1\,47$
14
1
———
$9\,)\,16 \cdot \dot{3}$
———
$1 \cdot 814,\dot{8}$,&c.

For

For $\cdot\dot{7} \times 2\cdot\dot{3} = \frac{\cdot 7 \times 10}{9} \times \frac{2\cdot 3 \times 10 - 2}{9}$ (Art. 9.) $= \frac{147}{9 \times 9}$ $= \frac{147}{9 \times 10 - 9}$. (Art. 12.)

And $\frac{147}{9 \times 9} = 147 \times \frac{1}{9} \times \frac{1}{9} = \frac{14\cdot 7 + 1\cdot 47 + \cdot 147, \text{\&c.}}{9}$

For the Learner's Exercise.

Multiply $\cdot\dot{4}$ by $2\cdot\dot{6}$; $\cdot\dot{8}$ by $\cdot 01\dot{4}$; $48\cdot\dot{7}$ by $\cdot\dot{4}$; $4\cdot 6\dot{8}$ by $\dot{9}$; $\cdot 24\dot{6}$ by $\dot{2}$; $48\cdot\dot{7}$ by $\cdot 0\dot{4}$.

2. A pure single circulate, and a mixed multiple circulate.

Example.

Multiply $4\dot{2}4\cdot\dot{3}$ by $\cdot 0\dot{3}$.

$$\begin{array}{r}
424\cdot 3 \\
1000 \\
\hline
424\ 300 \\
4 \\
\hline
423\ 900 \\
\cdot 3 \\
\hline
127\cdot 17 \\
127 \\
\hline
12\cdot 729 \\
1\ 272 \\
127 \\
12 \\
1 \\
\hline
1\dot{4}\cdot 1\dot{4}
\end{array}$$

Or thus,

$$\begin{array}{r}
4\dot{2}4\cdot\dot{3} \\
\cdot 3 \\
\hline
9)12\dot{7}\cdot 2\dot{9} \\
\hline
1\dot{4}\cdot 1\dot{4}
\end{array}$$

For $4\acute{2}4\cdot\acute{3}\times\cdot0\acute{3}=\frac{424\cdot3\times1000-400}{999}\times\frac{\cdot3}{9}=\frac{127170}{9\times999}$ $=127\cdot17+\cdot12717+\cdot000127$, &c. $\times\frac{1}{9}=12\cdot7\dot{2}\dot{9}+1\cdot2729$ $+\cdot12729+\cdot0127$, &c.

The reaſon of the other proceſs is evident.

For the Learner's Exerciſe.

Multiply $\cdot\acute{2}7\acute{6}$ by $\cdot\acute{6}$; $4\acute{8}\cdot5\acute{1}$ by $\cdot\acute{3}$; $7\cdot1\acute{0}\acute{4}$ by $\cdot0\acute{2}$; $\cdot\acute{6}\acute{8}$ by $\cdot\acute{1}$; $\cdot0\acute{1}$ by $\acute{2}\cdot\acute{4}6\acute{1}$; $\cdot0\acute{6}$ by $\acute{4}\cdot00\acute{2}$.

3. A pure multiple circulate, and a mixed ſingle circulate.

Example.

Multiply $\acute{2}6\cdot\acute{8}$ by $\cdot02\acute{6}$.

$$
\begin{array}{r}
\cdot026 \\
10 \\
\hline
\cdot26 \\
\cdot02 \\
\hline
\cdot24 \\
26800 \\
\hline
6\cdot432 \\
64. \\
\hline
9)6\cdot\acute{4}3\acute{8} \\
\hline
\cdot\acute{7}1538204\acute{8}
\end{array}
\qquad \text{Or thus,} \qquad
\begin{array}{r}
\acute{2}6\cdot\acute{8}\times\cdot02 \\
\cdot06 \\
\hline
9)1\cdot\acute{6}0\acute{9} \\
\hline
\cdot1\acute{7}8845\acute{5}12 \\
\cdot\acute{5}3\acute{6} \\
\hline
\cdot\acute{7}1538204\acute{8}.
\end{array}
$$

For

For $\dot{2}6\cdot\dot{8}\times\cdot 0\dot{2}\dot{6}=\frac{26\cdot 8\times 1000}{999}\times\frac{\cdot 026\times 10-\cdot 02}{9}=\frac{6432}{9\times 999}$
$=\frac{6\cdot 432+\cdot 0064,\ \&c.}{9}$.

And $\cdot 02\dot{6}=\cdot 02+\cdot 00\dot{6}=\cdot 02+\frac{\cdot 06}{9}$ (Art. 16. 7.) $\therefore\ \cdot 02\dot{6}\times\dot{2}\dot{6}\cdot\dot{8}$
$=\cdot 02\times\dot{2}6\cdot\dot{8}+\frac{\cdot 06\times\dot{2}6\cdot\dot{8}}{9}$.

For the Learner's Practice.

Multiply $\dot{4}\cdot 6\dot{8}$ by $4\cdot\dot{1}$; $\cdot\dot{4}\dot{1}$ by $6\cdot 8\dot{7}$; $\cdot 6\dot{8}\dot{1}$ by $\cdot 00\dot{4}\dot{6}$; $\cdot 4\dot{6}$ by $\dot{2}\cdot 4\dot{6}$; $\cdot 2\dot{7}\dot{8}$ by $\dot{4}\cdot\dot{6}$.

4. A pure multiple circulate, and a mixed multiple circulate.

Example.

Multiply $\cdot 04\dot{2}\dot{3}$ by $\cdot 0\dot{3}\dot{8}$.

	·0423	Or thus,	$\cdot 04\dot{2}\dot{3}$
	100		$3\cdot\dot{8}$
	4·23		
	·04		$33\dot{8}\dot{5}$
9900	4·19		$12\dot{6}\dot{9}$
99	3·8		
9801	)15·922(·001624528+		$\cdot 00160\dot{8}\dot{2}$, 82, 82, &c.
			160 82, 82, &c.
			1 60, 82, &c.
			1 60, &c.
			·0016245 28 +

 For

$$\text{For } \cdot04\dot{2}\dot{3} \times \cdot0\dot{3}\dot{8} = \frac{\cdot0423 \times 100 - \cdot04}{99} \times \frac{3\cdot8}{99} = \frac{15\cdot922}{99 \times 100 - 99}.$$

$$\text{And } \cdot0\dot{3}\dot{8} = 3\cdot8 \times \frac{1}{99} \therefore \cdot04\dot{2}\dot{3} \times 3\cdot8 \times \frac{1}{99} = \cdot00160\dot{8}\dot{2} +$$

$\cdot000016 0\dot{8}\dot{2}$ + &c.

For the Learner's Exercise.

Multiply $\dot{2}4\cdot\dot{8}$ by $6\cdot\dot{8}\dot{4}$; $\dot{6}8\cdot\dot{7}$ by $\dot{4}\dot{6}\cdot\dot{8}$; $\dot{2}\dot{1}\cdot8\dot{2}$ by $4\cdot\dot{6}\dot{8}$; $6\dot{8}\dot{2}\cdot\dot{1}$ by $\dot{4}\cdot\dot{6}$; $2\cdot\dot{1}\dot{9}$ by $\cdot\dot{4}1\dot{6}$.

CASE IV.

When the factors are both pure circulates.

RULE.

Multiply the numbers as terminates, and remove the decimal point as many places to the right-hand as there are figures in both repetends; then find a divisor, agreeable to the places of the repetend; or, add the products as before.

1. Two pure single circulates.

Example.

Multiply $\cdot\dot{8}$ by $\cdot0\dot{3}$.

$$\begin{array}{rl} & \cdot8 \\ 90 & \cdot03 \\ 9 & \overline{} \\ \hline 81)2\cdot4(& \cdot0\dot{2}9\dot{6} \end{array}$$

Or thus,

$$\begin{array}{r} \cdot\dot{8} \\ \cdot3 \\ \hline 9)\cdot2\dot{6} \\ \hline \cdot\dot{0}2\dot{9}6 \end{array}$$

For

$$\text{For } \cdot\dot{8} \times \cdot 0\dot{3} = \frac{\cdot 8 \times 10}{9} \times \frac{\cdot 03 \times 10}{9} = \frac{\cdot 8 \times \cdot 03 \times 100}{9 \times 9}.$$

$$\text{And } \cdot 0\dot{3} = \frac{\cdot 3}{9} \therefore \cdot\dot{8} \times \cdot 0\dot{3} = \frac{\cdot\dot{8} \times \cdot 3}{9}.$$

For the Learner's Exercise.

Multiply $\cdot\dot{8}$ by $\cdot\dot{4}$; $\cdot 0\dot{7}$ by $\cdot 0\dot{1}$; $\cdot\dot{4}$ by $\cdot\dot{2}$; $\cdot 06$ by $\cdot\dot{1}$; $\cdot 00\dot{4}$ by $\cdot\dot{5}$.

2. A pure single circulate, and a pure multiple circulate.

Example.

Multiply $421\cdot\dot{4}2\dot{1}$ by $\cdot 00\dot{2}$.

$$\begin{array}{l} 421\cdot \\ \cdot 002 \\ \hline 8\cdot 42 \\ \quad\ 84 \text{ \&c.} \\ \hline 9)\dot{8}\cdot 4\dot{2} \\ \hline \quad \cdot 93649\dot{2}04760\dot{3}. \end{array}$$

$$\text{For } 421\cdot\dot{4}2\dot{1} \times \cdot 00\dot{2} = \frac{421 \times 1000}{999} \times \frac{\cdot 002 \times 10}{9} = \frac{8420 \times \frac{1}{999}}{9}$$

$$= \frac{8\cdot 42 + \cdot 00842, \text{ \&c.}}{9}$$

For

For the Learner's Practice.

Multiply 68·2 by ·2; 17·6 by ·8; ·762 by ·04; ·68 by ·006; ·04 by ·683.

3. Two pure multiple circulates.

Example.

Multiply ·324 by 4·6.

```
   ·324        Or thus,        ·324
    4·6                         460
  ------                       -----
 149·04                        1945
     149..                    1297
 --------                     -----
1·491891                   99)149·18
   14918                      --------
     149                      1·50696.
       1
 --------
1·50696
```

For $\cdot 324 \times 4\cdot 6 = \frac{\cdot 324 \times 1000}{999} \times \frac{4\cdot 6 \times 100}{99} = \cdot 324 \times 4\cdot 6 \times 100000 \times \frac{1}{999} \times \frac{1}{99} = 1\cdot 4918 + \cdot 014918 + \cdot 00014$, &c.

And $4\cdot 6 = \frac{460}{99} \therefore \frac{\cdot 324 \times 460}{99} = 1\cdot 50696.$

For the Learner's Exercise.

Multiply ·871 by 4·1; 68·2 by ·014; ·601 by ·002; ·48 by ·048.

CASE

CASE V.

If both factors are mixed circulates.

RULE.

Multiply the numbers as terminates, and remove the decimal point as many places to the right hand as the number of figures in both repetends; to which add the product of the terminate parts, and reserve the sum. Multiply the terminate parts by the circulates reciprocally, having their decimal points brought forward according to their places, and subtract the sum of these products from the reserved sum: then proceed as before, by finding a divisor; or adding the remainders.

1. Two mixed single circulates.

Example.

Multiply $2{\cdot}1\dot{4}$ by $4{\cdot}\dot{3}$.

$$\begin{array}{rl}
 & 2{\cdot}14 \\
 & \underline{4{\cdot}3} \\
 & 920{\cdot}2 \\
4\times 2{\cdot}1 = & \underline{8{\cdot}4} \\
 & 928{\cdot}6 \\
 & \underline{175{\cdot}9} = 90{\cdot}3 + 85{\cdot}6 \\
 & 9)\underline{752{\cdot}7} \\
 & 9)\underline{\;83{\cdot}63} \\
 & 9{\cdot}2\dot{9}2\dot{5}
\end{array}
\qquad
\begin{array}{r}
43 \\
\underline{2{\cdot}1} \\
90{\cdot}3
\end{array}
\qquad
\begin{array}{r}
21{\cdot}4 \\
\underline{4} \\
85{\cdot}6
\end{array}$$

Or thus,

$$\begin{array}{r}
2{\cdot}1\dot{4}\times 4 \\
\underline{3} \\
9)\underline{6{\cdot}4\dot{3}} \\
{\cdot}7\dot{1}4\dot{8} \\
\underline{8{\cdot}5\dot{7}77} \\
9{\cdot}2\dot{9}2\dot{5}
\end{array}$$

For $2{\cdot}1\dot{4} \times 4{\cdot}\dot{3} = \dfrac{2{\cdot}14\times 10 - 2{\cdot}1}{9} \times \dfrac{4{\cdot}3\times 10 - 4}{9} =$

$$\frac{2{\cdot}14\times 4{\cdot}3\times 100 + 2{\cdot}1\times 4 - \overline{2{\cdot}1\times 4{\cdot}3\times 10 + 4\times 2{\cdot}1\times 10}}{9\times 9}.$$

And

And $4\cdot\acute{3} = 4 + \frac{3}{9} \therefore 2\cdot1\acute{4} \times 4\,\acute{3} = 4 \times 2\cdot1\acute{4} + \frac{3 \times 2\cdot1\acute{4}}{9}$.

For the Learner's Exercise.

Multiply $2\cdot\acute{8}$ by $41\cdot\acute{6}$; $6\cdot8\acute{7}$ by $\cdot02\acute{1}$; $\cdot6\acute{1}$ by $4\cdot8\acute{7}$; $4\cdot8\acute{9}$ by $6.\acute{4}$; $\cdot4\acute{6}$ by $\cdot01\acute{7}$.

2. A mixed single circulate, and a mixed multiple circulate.

Example.

Multiply $\cdot01\acute{7}$ by $1\cdot\acute{0}0\acute{3}$.

$$\begin{array}{r}
1\cdot003 \\
\cdot017 \\
\hline
170\cdot51 \\
\cdot01 \\
\hline
170\cdot52 \\
10\cdot2 \\
\hline
\cdot16032 \\
160 \\
\hline
9)\cdot160\acute{4}\acute{8} \\
\hline
\cdot017\acute{8}311644\acute{9}
\end{array}$$

Or thus,

$$\begin{array}{r}
1\cdot\acute{0}0\acute{3} \times \cdot01 \\
\cdot07 \\
\hline
9)\cdot070\acute{2}\acute{1} \\
\hline
\cdot00780113446 \text{ \&c.} \\
\cdot01003003003 \\
\hline
\cdot017\acute{8}311644\acute{9}
\end{array}$$

For $\cdot01\acute{7} \times 1\cdot\acute{0}0\acute{3} = \frac{\cdot017 \times 10 - \cdot01}{9} \times \frac{1\cdot003 \times 1000 - 1}{999} = \frac{170\cdot51 + \cdot01 - 10\cdot2 \times \frac{1}{999}}{9}$.

And $\cdot01\acute{7} = \cdot01 + \frac{\cdot07}{9}$.

For

For the Learner's Exercise.

Multiply $46\cdot\dot{8}$ by $4\dot{6}\cdot\dot{8}$; $\cdot21\dot{8}$ by $4\cdot\dot{6}\dot{8}$; $26\cdot\dot{4}$ by $\cdot\dot{9}02\dot{4}$; $7\dot{1}\cdot\dot{6}$ by $\cdot4\dot{6}$; $4\dot{8}\cdot9\dot{1}$ by $6\cdot82\dot{7}$.

3. Two mixed multiple circulates.

Example.

Multiply $\cdot01\dot{3}\dot{2}$ by $\dot{2}0\cdot\dot{1}$.

$$\begin{array}{rr} & \cdot0132 \\ & 20\cdot1 \\ \hline & 2653\cdot2 \\ & \cdot2 \\ \hline 9900 & 2653\cdot4 \\ 99 & 46\cdot5 \\ \hline 9801 &)2606\cdot9(\cdot26598306+ \end{array}$$

Or thus,

$$\begin{array}{r} \dot{2}0\cdot\dot{1}\times\cdot01 \\ \cdot32 \\ \hline \cdot0643\dot{2},32,32, \\ 64,32,32, \\ 64,32 \\ 64 \\ \hline \cdot06497296 \\ \cdot20101010 \\ \hline \cdot26598306+ \end{array}$$

For $\cdot01\dot{3}\dot{2}\times\dot{2}0\cdot\dot{1}=\dfrac{\cdot0132\times100-\cdot01}{99}\times\dfrac{20\cdot1\times100-20}{99}$

$=\dfrac{2606\cdot9}{99\times100-99}$.

And $\cdot01\dot{3}\dot{2}=\cdot01+\cdot00\dot{3}\dot{2}=\cdot01+\dfrac{32}{99}$.

For the Learner's Exercise.

Multiply $1\dot{4}\cdot6\dot{8}$ by $2\cdot\dot{6}\dot{8}$; $\cdot1\dot{6}\dot{8}$ by $\cdot01\dot{4}\dot{6}$; $6\dot{8}7\cdot1\dot{2}$ by $\cdot4\dot{8}7\dot{1}$; $\cdot\dot{6}871\cdot9\dot{6}$ by $68\dot{7}19\cdot\dot{5}$.

SECTION

SECTION V.

Division of Circulates.

CASE I.

When the dividend is a circulate (pure or mixed) and the divisor a terminate.

RULE.

PROCEED as in common division, continuing the repeating figures in the dividend (if necessary) till the quotient repeats or terminates.

Example 1.

Divide $\cdot\dot{4}2\dot{7}$ by 8.

$8)\cdot 427,427,4$ &c.$(.053\dot{4}2\dot{8},4$ &c.

Example 2.

Divide $2\cdot 1\dot{6}8\dot{4}$ by $68\cdot 7$.

$68\cdot 7)2\cdot 1684,684,6$ &c.$(\cdot 0315641$ &c.

Examples for the Learner's Exercise.

Divide $\dot{6}\cdot 8\dot{7}$ by 27; $\cdot 2\dot{4}\dot{1}$ by $8\cdot 1$; $68\cdot\dot{4}\dot{1}$ by $\cdot 091$; $\dot{6}\cdot 4\dot{1}$ by $\cdot 06$; $\cdot 02\dot{1}\dot{4}$ by 48.

CASE

CASE II.

When the dividend is a terminate, and the divisor a pure circulate.

RULE.

Multiply the terminate by as many *nines* as there are repeating figures in the circulate for a new dividend, and remove the decimal point in the circulate the same places to the right for a new divisor; then proceed as in common division, continuing the quotient (if necessary) till it repeats or terminates.

Example.

Divide 7 by $\cdot\dot{6}$.

$$\begin{array}{r} 7 \\ 9 \\ \hline 6)63(10{\cdot}5 \end{array}$$

For $\cdot\dot{6} = \frac{6}{9}$, and $7 \div \frac{6}{9} = \frac{9}{6} \times \frac{7}{1} = \frac{63}{6} = 10{\cdot}5$.

For the Learner's Practice.

Divide 4 by $\cdot\dot{6}$; $\cdot 8$ by $2{\cdot}\dot{2}$; $\cdot 6$ by $\cdot 0\dot{8}$; $\cdot 5$ by $\cdot 0\dot{8}$; $\cdot 06$ by $\cdot\dot{4}$; $\cdot 007$ by $\dot{2}$.

2. A terminate consisting of several figures, and a pure single circulate.

Example.

Divide 43·2 by $\cdot\dot{7}$.

$$\begin{array}{r} 43{\cdot}2 \\ 9 \\ \hline 7)388{\cdot}8(55{\cdot}5\dot{4}2857\dot{1} \end{array}$$

For

For $\cdot\dot{7} = \frac{7}{9}$, and $43\cdot2 \div \frac{7}{9} = \frac{9}{7} \times \frac{43\cdot2}{1} = \frac{388\cdot8}{7}$ $= 55\cdot54$, &c.

For the Learner's Practice.

Divide 764 by $\cdot\dot{3}$; $\cdot68$ by $8\cdot\dot{8}$; $\cdot2816$ by $\cdot00\dot{6}$; $2\cdot2$ by $\cdot00\dot{7}$; $\cdot087$ by $\cdot\dot{5}$.

3. A single terminate, and a pure multiple circulate.

Example.

Divide $\cdot6$ by $2\cdot\dot{3}\dot{1}$.

$$\begin{array}{r} \cdot6 \\ 999 \\ \hline 2310)599\cdot4(\cdot25948\text{, \&c.} \end{array}$$

For $2\cdot\dot{3}\dot{1} = \frac{2310}{999}$, and $\cdot6 \div \frac{2310}{999} = \frac{999}{2310} \times \cdot6$ $= \cdot25948$, &c.

For the Learner's Exercise.

Divide 8 by $2\dot{3}\cdot\dot{1}$; $\cdot7$ by $\dot{8}7\cdot1\dot{6}$; $\cdot08$ by $\cdot\dot{0}8\dot{7}$; 4 by $619\cdot\dot{6}1\dot{9}$; $\cdot009$ by $\cdot00\dot{1}\dot{7}$.

4. A terminate of several figures, and a pure multiple circulate.

Example.

Divide $48\cdot76$ by $\dot{3}2\cdot4\dot{1}$.

$$\begin{array}{r} 487600 \\ 48\cdot76 \\ \hline 324100)487551\cdot24(1\cdot5043\text{, \&c.} \end{array}$$

For

For $\dot{3}2{\cdot}4\dot{1} = \frac{324100}{9999}$, and $48{\cdot}76 \div \frac{324100}{9999} = \frac{9999 \times 48{\cdot}76}{324100}$

$= \frac{48{\cdot}76 \times 10000 - 48{\cdot}76}{324100}$.

For the Learner's Exercise.

Divide, 3871 by $\dot{4}6{\cdot}\dot{8}$; 48·71 by $\dot{7}1{\cdot}\dot{6}$; ·0087 by $\dot{2}{\cdot}6\dot{8}\dot{7}$; $487{\cdot}1$ by $\dot{\cdot}68\dot{7}\dot{1}$; $271.2\dot{7}\dot{1}$.

CASE III.

If the dividend be a terminate, and the divisor a mixed circulate.

RULE.

Multiply the terminate by as many *nines* as there are places in the repetend for a new dividend; then remove the decimal point in the circulate the same places to the right-hand, from which subtract the given circulate, and the remainder will be the new divisor, with which proceed as in common division.

1. A single terminate, and a mixed single circulate.

Example.

Divide 8 by $3{\cdot}2\dot{7}$.

$$\begin{array}{rr}
3{\cdot}27 & \\
32{\cdot}77 & 8 \\
3{\cdot}27 & 9 \\
\hline
29{\cdot}5\ )72(2{\cdot}44067, \&c. &
\end{array}$$

For $3{\cdot}2\dot{7} = \frac{3{\cdot}27 \times 10 - 3{\cdot}27}{9} = \frac{29{\cdot}5}{9}$, and $\frac{8}{1} \div \frac{29{\cdot}5}{9} =$

$\frac{9}{29{\cdot}5} \times 8 = \frac{72}{29{\cdot}5}$.

For the Learner's Practice.

Divide, 7 by 48·7́; ·4 by 48·2́; ·02 by 2·7́; ·008 by ·087́; ·007́ by 6·87́; 8 by 46.9́.

2. A single terminate, and a mixed multiple circulate.

Example.

Divide ·6 by 3·27́4́.

```
 3·27́4́
327·474 &c.    6
  3·274 &c.   99
_________   ____
 324·2      )59·4(·18321 &c.
```

For $3{\cdot}2\acute{7}\acute{4} = \frac{324{\cdot}2}{99}$, $\therefore {\cdot}6 \div \frac{324{\cdot}2}{99} = \frac{99 \times 6}{324{\cdot}2}$

For the Learner's Practice.

Divide 4 by 2·16́8́; ·8 by 66́·81́; ·7́ by ·48́7́; 6 by ·21́9́; ·04 by 58́1·901́.

3. A terminate of several figures, and a mixed single circulate.

Example.

Divide 4·56 by 3·14.

```
 3·14
31·4   4·56
 3·1      9
____   ____
28·3 )41·04(1·45017, &c.
```

For

For $3\cdot1\dot{4} = \frac{28\cdot3}{9}$, ∵ $4\cdot56 \div \frac{28\cdot3}{9} = \frac{4\cdot56 \times 9}{28\cdot3}$.

For the Learner's Practice.

Divide, $\cdot768$ by $2\cdot1\dot{4}$; $2\cdot07$ by $2\cdot8\dot{7}$; $\cdot0681$ by $21\cdot4\dot{6}$; $\cdot716$ by $\cdot41\dot{6}$; $\cdot0081$ by $\cdot08\dot{7}\dot{4}$.

4. A terminate of several figures, and a mixed multiple circulate.

Example.

Divide $13\cdot271$ by $4\dot{7}\cdot68\dot{4}$.

```
47·684
47 6847·6   132710.
     47·6     13·271
---------   --------
476800.   )132696·729(·2783, &c.
```

For $4\dot{7}\cdot68\dot{4} = \frac{476800}{9999}$, ∵ $13\cdot2171 \div \frac{476800}{9999} = \frac{9999 \times 13\cdot271}{476800} = \frac{13\cdot271 \times 10000 - 13\cdot271}{476800}$.

For the Learner's Exercise.

Divide, $2\cdot714$ by $28\cdot\dot{4}1\dot{6}$; $3\cdot871$ by $4\cdot\dot{8}7\dot{1}$; $\cdot0487$ by $4\cdot\dot{8}\dot{1}$; $\cdot487$ by $\cdot9\dot{1}6\dot{8}$; $46\cdot87$ by $2\dot{1}\cdot9\dot{1}$.

CASE IV.

If the dividend be a pure circulate, and the divisor a mixed circulate.

RULE.

Remove the decimal point of the dividend as many places to the right-hand as repeating figures in both numbers, from which subtract the said dividend, punctuated according to its place, for a new dividend. Then multiply the divisor by an an *unit* with as many *cyphers* annexed as its repetend hath places; this product lessened by its terminate part, and the decimal point removed to the right, according to the places of the first dividend, and again lessened by the last remainder, will give the new divisor.

1. A pure single circulate, and a mixed single circulate.

Example.

Divide $\dot{\cdot 6}$ by $2\cdot\dot{3}$.

$2\cdot 3$ Or thus, $21)6.(\cdot 28571$, &c.
10
———
23
2
—
21 $\cdot\dot{6}$
210 60
21 6
——— —
189)54($\cdot 28571$, &c.

For

For $\cdot\dot{6} = \frac{6}{9}$, and $2\cdot\dot{3} = \frac{2\cdot\dot{3}\times10-2\cdot\dot{3}}{9} = \frac{21}{9}$, $\therefore \frac{6}{9} \div \frac{21}{9} =$ $\frac{9}{21} \times \frac{6}{9} = \frac{6\times10-6}{2\cdot3\times10-2\times10-2\cdot3\times10-2} = \frac{6}{21}$.

For the Learner's Exercise.

Divide $\cdot\dot{4}$ by $2\cdot\dot{6}$; $\cdot\dot{8}$ by $46\cdot\dot{8}$; $\cdot0\dot{8}$ by $\cdot6\dot{8}\dot{7}$; $\cdot\dot{7}$ by $6\dot{8}\cdot4\dot{1}$; $\cdot9\dot{8}$ by $46\cdot9\dot{8}$.

2. A pure single circulate, and mixed multiple circulate.

Example.

Divide $\cdot0\dot{7}$ by $6\dot{3}1\cdot\dot{4}$.

$$
\begin{array}{r l}
6\dot{3}1\cdot\dot{4} & \\
100 & \\
\hline
63140 & \\
6 & \\
\hline
62540 & \cdot07 \\
625400 & 700 \\
62540 & \cdot7 \\
\hline
562860 &)\ 699\cdot3(\cdot00124\ \&c.
\end{array}
$$

For $\cdot9\dot{7} \div 6\dot{3}1\cdot\dot{4} = \frac{\cdot0\dot{7}\times10000-\cdot07\times10}{6\dot{3}1\cdot\dot{4}\times100-6\times10-6\dot{3}1\cdot\dot{4}\times100-6}$ $= \frac{699\cdot3}{562860}$.

For the Learner's Practice.

Divide $\cdot\dot{4}$ by $27\cdot\dot{1}\dot{4}$; $7\cdot\dot{7}$ by $\cdot4\dot{8}7\dot{1}$; $\cdot00\dot{8}$ by $3\dot{8}\cdot2\dot{6}$; $6\cdot\dot{6}$ by $\cdot\dot{2}1\dot{6}$; $11\cdot\dot{1}$ by $4\cdot\dot{8}18\dot{7}$.

3. A pure multiple circulate, and a mixed single circulate.

Example.

Divide $\dot{3}4\cdot\dot{6}$ by $\cdot04\dot{3}$.

$$
\begin{array}{ll}
\cdot04\dot{3} & \\
\quad 10 & \\
\hline
\cdot43 & \\
\cdot04 & \\
\hline
\cdot39 & \dot{3}4\cdot\dot{6} \\
390 & 346000 \\
\quad \cdot39 & \quad 34600 \\
\hline
389\cdot61 &)\,311400(799\cdot26079,\ \&c.
\end{array}
$$

For $\dot{3}4\cdot\dot{6}\div\cdot04\dot{3}=\dfrac{\dot{3}4\cdot\dot{6}\times10000-\dot{3}4\cdot6\times1000}{\overline{\cdot04\dot{3}\times10-\cdot04}\times1000-\overline{\cdot04\dot{3}\times10-\cdot04}}$

$=\dfrac{311400}{389\cdot61}$.

For the Learner's Practice.

Divide $2\cdot\dot{4}\dot{1}$ by $7\cdot\dot{1}\dot{4}$; $2\cdot\dot{4}\dot{8}$ by $\cdot27\dot{1}$; $\dot{4}1\cdot6\dot{8}$ by $\cdot0\dot{6}\dot{4}$; $\cdot\dot{4}8\dot{7}$ by $6\dot{8}\cdot\dot{9}$; $\dot{4}6\cdot\dot{8}$ by $48\cdot\dot{8}$.

4. A pure multiple circulate, and a mixed multiple circulate.

Example.

Example.

Divide $\cdot 0\dot{4}\dot{2}$ by $.03\dot{2}\dot{6}$.

$\cdot 03\dot{2}\dot{6}$ Or thus, $3\cdot 23)4\cdot 2(1\cdot 3003$, &c.

$$\begin{array}{rr}
\cdot 03\dot{2}\dot{6} & \\
100 & \\
\hline
3\cdot 26 & \\
\cdot 03 & \\
\hline
3\cdot 23 & \cdot 042 \\
3\,23 & 420 \\
3\cdot 23 & 4\cdot 2 \\
\hline
319\cdot 77\ ) & 415\cdot 8(1\cdot 3003,\ \&c.
\end{array}$$

For $\cdot 0\dot{4}\dot{2} \div \cdot 03\dot{2}\dot{6} = \frac{\cdot 042 \times 100}{99} \div \frac{\cdot 0326 \times 100 - \cdot 03}{99} = \frac{4\cdot 2}{3\cdot 23}$.

For the Learner's Exercise.

Divide $\dot{4}\cdot\dot{2}6$ by $4\dot{2}\cdot\dot{7}6$; $4\dot{1}\cdot\dot{6}$ by $\cdot 07\dot{1}\dot{4}$; $\cdot 0\dot{8}\dot{7}$ by $\cdot 005\dot{6}\dot{8}$; $\dot{2}\dot{6}\cdot 8$ by $4\cdot\dot{1}\dot{7}$; $\dot{6}0\cdot\dot{0}1$ by $.01\dot{7}\dot{5}6$.

Remark 1.

It appears from the first and last Examples, that when the places of repeating figures in both numbers are equal, the work may be contracted, by rejecting the denominators in the finite values of the given circulates.

2. It is also evident that this rule holds good (*mutatis mutandis*) when the *dividend* is a mixed circulate, and the *divisor* a pure circulate.

Example.

Divide $4{\cdot}20\dot{7}\dot{6}$ by $\dot{3}4{\cdot}1\dot{2}$.

$$
\begin{array}{rr}
 & 4{\cdot}20\dot{7}\dot{6} \\
 & 100 \\
\hline
 & 420{\cdot}76 \\
 & 4{\cdot}2 \\
\hline
34{\cdot}12 & 416{\cdot}56 \\
34\ 120000 & 416\ 5600 \\
341200 & 416{\cdot}56 \\
\hline
33778800 &)\ 4165183{\cdot}44(\cdot 1233, \&c.
\end{array}
$$

For $4{\cdot}20\dot{7}\dot{6} \div \dot{3}4{\cdot}1\dot{2} = \dfrac{\overline{4{\cdot}2076 \times 100 - 4{\cdot}20} \times 10000 - \overline{4{\cdot}2076 \times 100 - 4{\cdot}2}}{34{\cdot}12 \times 1000000 - 34{\cdot}12 \times 1000}$.

For the Learner's Practice.

Divide $4{\cdot}\dot{2}\dot{7}$ by $\dot{2}{\cdot}1\dot{6}$; $8\dot{4}{\cdot}7\dot{1}$ by $\dot{5}01{\cdot}\dot{6}$; $4{\cdot}\dot{8}7\dot{1}$ by $\dot{4}0{\cdot}1\dot{7}$; $\cdot 6\dot{2}1\dot{0}$ by $\dot{5}07{\cdot}\dot{6}$; $\cdot 001\dot{7}\dot{6}$ by $\cdot 0\dot{1}4\dot{6}$; $\cdot 007\dot{4}\dot{3}$ by $\cdot 001\dot{7}1\dot{6}$.

CASE V.

When both the dividend and divisor are pure circulates.

RULE.

Find a new dividend and a new divisor, by removing the decimal points as many places to the right-hand as there are repeating figures in both numbers, and then subtracting the numerators of their respective terminate values.

1. Two pure single circulates.

Example.

Example.

Divide $\cdot\dot{6}$ by $\cdot 0\dot{4}$.

$$\begin{array}{rr} \cdot 0\dot{4} & \cdot\dot{6} \\ 4 & 60 \\ \cdot 4 & \cdot 6 \\ \hline 3\cdot 6 &)54(15. \end{array} \qquad \text{Or,} \qquad \cdot 4)6(15$$

For $\cdot\dot{6} = \frac{6}{9}$, and $\cdot 0\dot{4} = \frac{\cdot 4}{9}$, $\therefore \frac{6}{9} \div \frac{\cdot 4}{9} = \frac{\cdot 6 \times 100 - 6}{\cdot 04 \times 100 - 4}$ $= \frac{\cdot 6}{\cdot 4}$.

For the Learner's Practice.

Divide $\cdot\dot{7}$ by $\cdot\dot{8}$; $\cdot 0\dot{4}$ by $2\cdot\dot{2}$; $7\cdot\dot{7}$ by $\cdot\dot{6}$; $\cdot 0\dot{1}$ by $4\cdot\dot{4}$; $\cdot\dot{8}$ by $\cdot 0\dot{7}$; $\cdot 00\dot{8}$ by $\cdot 000\dot{4}$.

2. Two pure multiple circulates.

Example.

Divide $\cdot\dot{4}3\dot{2}$ by $\dot{2}\cdot\dot{3}$.

$$\begin{array}{rr} 2\cdot 3 & \cdot 432 \\ 230000 & 43200 \\ 230 & 432 \\ \hline 229770 &)42768(\cdot 18526, \&c. \end{array}$$

For the Learner's Exercise.

Divide $\dot{4}6\cdot\dot{8}$ by $\dot{6}\cdot\dot{7}$; $\cdot\dot{5}0\dot{4}$ by $\dot{1}\cdot 6\dot{8}$; $\cdot 0\dot{1}\dot{4}$ by $\dot{7}\cdot 6\dot{5}$; $\dot{2}1\cdot\dot{8}$ by $\cdot 00\dot{1}\dot{4}$; $\cdot 0\dot{8}1\dot{9}$ by $\cdot 0\dot{7}\dot{1}$.

3. A pure single circulate, and a pure multiple circulate.

Example.

Example 1.

Divide ·004 by 341·341

341	·004
3410000	40
341000	·04
3069000)	39·96(·00001302, &c.

Example 2.

Divide 70·046 by 4·4.

4·4	
4000000	70046000
40	7004600
3999960)	63041400(15·7605, &c.

For the Learner's Practice.

Divide ·4 by ·682; ·08 by ·427; ·001 by 5·04; ·0871 by ·7; 4·81 by ·04.

CASE VI.

If the dividend and divisor are both mixed circulates.

RULE.

Find the new divisor and dividend by removing the decimal point of the terminate value of each numerator as many places to the right-hand as there are repeating figures in the other number, and then subtracting their respective numerators.

1. Two mixed single circulates.

Example.

Example.

Divide $21{\cdot}\acute{4}$ by $3{\cdot}\acute{4}$.

3·4	21·4	Or thus,	31(193(6·2258, &c.
34	214		
3	21		
310	1930		
31	193		
279	)1737(6·2258, &c.		

For $21{\cdot}\acute{4} \div 3{\cdot}\acute{4} = \frac{9}{3{\cdot}4 \times 10 - 3} \times \frac{21{\cdot}4 \times 10 - 21}{9} = \frac{193 \times 10 - 193}{31 \times 10 - 31} = \frac{193}{31}$.

For the Learner's Practice.

Divide $\acute{4}1{\cdot}\acute{7}$ by $8{\cdot}\acute{1}\acute{7}$; $98{\cdot}\acute{4}$ by ${\cdot}41\acute{8}$; ${\cdot}12\acute{7}$ by ${\cdot}04\acute{6}$; $1{\cdot}0\acute{4}$ by ${\cdot}08\acute{7}$; ${\cdot}71\acute{4}$ by ${\cdot}008\acute{4}$; ${\cdot}\acute{6}8\acute{7}$ by $99{\cdot}\acute{9}$.

2. Two mixed multiple circulates.

Example.

Divide ${\cdot}01\acute{3}\acute{2}$ by $3\acute{2}{\cdot}0\acute{1}$.

31980	1·31
3198000	1 310
31980	1·31
3166020	)1308·69(·0004133, &c.

For

For the Learner's Exercise.

Divide 4·871 by ·618; 4·109 by ·7181; ·4271 by ·0714; 5·071 by ·00714; ·4876 by ·01436; ·098714 by 48·71659.

3. A mixed single circulate and a mixed multiple circulate.

Example 1.

Divide ·016 by 1·014.

```
 1013     ·15
10130    150.
  1013     ·15
-------  -------
 9117 )149·85(·01643, &c.
```

Example 2.

Divide 21·07 by 4·21.

```
37·9       21050
37900      210500
    37·9    21050
--------  --------
37862·1 )189450(5·0034, &c.
```

For the Learner's Exercise.

Divide 4·16 by 2·184; 21·87 by 4·106; 187·19 by ·0176; ·4871 by 50·1; 201·47 by ·4071.

SECTION

SECTION VI.

Of the Logarithms of Repeating Decimals.

HAVING, in the foregoing Sections, supplied the pupil with rules for managing the whole doctrine of circulating numbers by common Arithmetic, and given the theory and reasons for the same, we shall therefore now proceed to shew how the whole business may be easily performed *logarithmically*, having first premised the following

LEMMA.

When one number is to be divided by another, the quotient will be the same as if *unity* were divided by the latter number, and the quotient multiplied by the former. For $\frac{A}{B} = \frac{1 \times A}{B} = \frac{1}{B} \times A.$

From hence, and the nature of Logarithms, it is evident, that $L.\frac{A}{B} = L.A - L.B = L.A + \overline{L.1 - L.B}$; and since in the terminate value of any circulating number, the denominator consists of as many *nines* as figures in the repetend, we shall evidently have this general rule for the logarithm of any circulate, pure or mixed.

RULE.

To the log. of the numerator of the terminate value of the given circulate add the arithmetical complement of as many *nines* as the repetend has places, and the sum will be the logarithm of the given circulate.

Example

A Table of the nine Digits perpetually circulating, and of the Arithmetical Complements of the Denominators.

Number.		Logarithm.	Number.		Logarithm.
$\dot{1}$	=	0·0457574	$\frac{1}{9}$	=	·9·0457574
$\dot{2}$	=	0·3467874	$\frac{1}{99}$	=	·8·0043647
$\dot{3}$	=	0·5228786	$\frac{1}{999}$	=	·7·0004344
$\dot{4}$	=	0·6478174	$\frac{1}{9999}$	=	·6·0000433
$\dot{5}$	=	0·7447274	$\frac{1}{99999}$	=	·5·0000043
$\dot{6}$	=	0·8239086	$\frac{1}{999999}$	=	·4·0000004
$\dot{7}$	=	0·8908554	$\frac{1}{9999999}$	=	·3·0000000
$\dot{8}$	=	0·9488474			
$\dot{9}$	=	1·0000000			

PRAC-

PRACTICAL QUESTIONS,

ADAPTED TO THE

PRECEDING RULES.

Question I.

WHAT is the Product of 14 feet 8 inches, by 11 feet 10 inches?

That is $14{\cdot}\dot{6} \times 11{\cdot}8\dot{3} = \frac{106{\cdot}5}{9} \times \frac{132}{9} = 173{\cdot}\dot{5} = 173$ f. 6 i. 8 p.

Question II.

Required the Area of 18 f. 9 i. 10 p. by 11 i. 9 p.?

That is $\frac{169{\cdot}375}{9} \times \frac{8{\cdot}8125}{9} =$ 18 f. 5 i. 1 p. 6 th. 6 fo.

Question III.

There is a Hall, the length of which is 96 f. 7 ⅔ i. and the breadth 68 f. 8 ⅓ i. which is to be paved with marble squares, each 1 f. 1 ½ i. squ. How many such squares will it take, and what does it come to at 2s 6d ½ *per* foot?

Question IV.

If 1 Cwt. 2 qrs. 18 lb. of Sugar cost 5l 1s 8d. what will 12 Cwt. 3 qrs. cost at the same rate?

Question V.

What must I give for 8 lb of Tobacco, when ¼ Cwt. cost 4l 17s 8d?

Question VI.

If the Penny-loaf weigh 7 ¾ Ounces, when Wheat is at 6s 4d *per* Bushel, what must be the weight of the Penny-loaf when Wheat is at 3s 10d *per* Bushel?

Question VII.

Suppose 4 Hhds, 3 Firkins, and 5 Gallons of Beer, cost 6l 14s 8d. How much is that *per* Hhd, and *per* Gn.?

Question VIII.

A piece of Land 4 rods broad and 40 long, being a Statute-Acre; it is required to know what length, with 10 rods and 2 yards breadth, will make an Acre?

Question IX.

A Bill of Exchange was accepted at London for the Payment of £847'53, for the same value delivered at Lisbon in Millrees; Exchange at 5s 4d *per* piece. How many Millrees were paid at Lisbon?

Question X.

A Merchant is desirous to know how much of each of the following Wines he must take, so that the whole Quantity may be 84 ⅝ Gns. at 5s 10d *per* Gn. Malaga at 7s 6d. Canary at 6s 9d. Sherry at 5s. and White Wine at 4s 3d. *per* Gn.

Question XI.

In what time will £7 *per Annum* pay a Debt of 120l 8s at 6 *per Cent.* Simple Interest?

Question

Queſtion XII.

Suppoſe a Freehold Eſtate of £40 $\frac{2}{3}$ *per Ann.* is to be ſold; what is it worth, allowing the Buyer £5 *per Cent.* Comp. Int. for his Money?

Queſtion XIII.

What is the Diſcount of 87l 13s 4d for 233 days, at £4 *per Cent. per Ann.*?

Queſtion XIV.

A challenges B to run a Race with him, if he will give him 3 rods in 10; now the velocity of B's running to that of A, is as 31 to 22. Which of the two beat?

Queſtion XV.

In the Year of our Lord 1775, the Cycle of the Sun was 9, and the Cycle of the Moon 20. Required from hence the Year of the Dionyſian Period.

Queſtion XVI.

Required the Area of the Parabola, whoſe Ordinate and Abſciſſa are $60\cdot\acute{4}\acute{2}$ and $4\acute{1}\cdot2\acute{4}$ reſpectively.

Queſtion XVII.

Required the Solidity of a Spheroid, of which the greater Axis is 12 $\frac{2}{3}$, and the leſs $6\cdot4\acute{2}\acute{7}$.

Queſtion XVIII.

Required the Quadrature of Hippocrates' Lunes, C. D. (Fig. 1.) the Diameter A B being 18 $\frac{1}{6}$, and A E 14 $\frac{2}{3}$.

Question XIX.

What is the Solidity of a Parabolic Conoid, the Diameter of the Base being 9·75, and the Height 11·25?

Question XX.

How many Solid Inches are contained in an Icosahedron, the side of which is 2·4 Inches?

MISCELLANEOUS QUESTIONS.

Question XXI.

"There are five whole numbers, the three first of which are in geometrical progression, and the three last in arithmetical, the second being the common difference in the last three. Now the sum of the four last $=$ 102, and the product of the second and last number is 504. Required the numbers."

Answer. Let v, w, x, y, z, represent the five numbers; then, *per Quest.* $vx = w^2$, $x + w = y$, $x + 2w = z$, $w + x + y + z = 102$, and $wz = 504$. For y and z substitute their values in the 4th. Equat. and it becomes $4w + 3x = 102$. And from the 3d and 5th Equat. we find $x = \frac{504}{w} - 2w$; this value of x being substituted in the last expression, gives $4w + \frac{1512}{w} - 6w = 102$, reduced, $w^2 + 51w = 756$, hence $w = 12$, and the required numbers are 8, 12, 18, 30, and 42.

Question XXII.

"To find the least whole number, which, being divided by 19, shall leave 17; if divided by 28, shall leave 21 remainder; and if the sum of the two resulting quotients be

subtracted

ſubtracted from the number ſought, the remainder, being divided by 3, ſhall leave 2 remaining."

Anſwer. The leaſt whole number that can poſſibly ſatisfy the two firſt conditions of the Queſtion is 245. *per* Simpſon's Algebra, *p*. 289. From whence likewiſe it is plain that the number ſought may be repreſented by $532x + 245$. But in order to obtain a general expreſſion for the ſum of the reſulting quotients, we muſt obſerve, that as 532 is a multiple of the Diviſors 19 and 28, any multiple (x) of the ſaid number will likewiſe contain a like multiple of theſe diviſors; and 245 being conſtant, the quotients 12 and 8 will be ſo too. Therefore $\overline{19+28} \times x + 20$ will truly repreſent the ſum of the quotients, which being ſubtracted from the number ſought leaves $485x+225$. Now 2 being taken from this number, and the remainder divided by 3, the quotients $\frac{485x+223}{3}$ will be a whole number by the queſtion, from whence the leaſt value of x will be found $= 1$, and conſequently that of $532x+245 = 777$; which is the number required.

Note. As $47x+20$ is a general expreſſion for the ſum of the quotients, any other diviſor and remainder beſides 3 and 2 may be propoſed, and the number anſwering the conditions of the queſtion found as above.

Queſtion XXIII.

"Required the dimenſions of the greateſt cylinder that can be inſcribed in a ſolid, formed by the rotation of a curve round its axis, whoſe equation is $ax^2 - x = y^4$, abſciſs 40, and ſemiordinate 28 inches?"

Anſwer. The greateſt cylinder that can be inſcribed in a ſolid, generated by a curve revolving about its axis, is when

the height of the cylinder is equal to half the ſubtangent (*per* Emerſon's Conics, *p.* 53). From the equation of the curve we have $\frac{4ax^2-4x}{2ax-1}$ = the ſubtangent; therefore $\frac{4ax^2-4x}{4ax-2}$ $=40-x$, reduced $x=\sqrt{\left(\frac{160a+6}{16a}\right)^2-\frac{10}{a}}+\frac{160a+6}{16a}=$ 19·9988. Therefore 40—19·9988 = 20·0011 = the height of the greateſt inſcribed cylinder; from whence the diameter is eaſily found by the equation of the curve.

Note. Had the equation of the curve been $ax^2-2x=y^4$, then the expreſſion for the ſubtangent $\frac{4ax^2-4x}{2ax-2}$ would be evidently $=2x$ (which is the property of the common or Apollonian Parabola) and the height of the cylinder = 20 exactly.

Queſtion XXIV.

"If in a plane triangle a right line be drawn from the vertical angle to the baſe, forming an angle at the ſame equal to the complement of half that at the vertex; the line ſo drawn will divide the difference of the ſegments of the baſe, in the ratio of the ſides, including the vertical angle: required the Demonſtration."

Anſwer. Draw BD (fig. B) making AD = the difference of the ſegments; and make BEC = the complement of half the vertical angle ABC. Let BF be taken = BC; and CF be drawn, and likewiſe AH ∥ to DB. Then it is evident becauſe of parallels, that the angle HAB = ABD = BCA—BAC = the Difference of the angles at the baſe. And in the triangles EGC, FGB, the angle FGB = EGC; and BED = BFG *per* conſtruc. therefore GCE = FBG. But (*per* Simpſon's Trig. *p.* 62.) the angle GCE = half the difference of the angles

angles at the base; consequently BE bisects the angle ABD, whence AB : BD (=BC) :: AE : ED (*per* Euc. 3, 6.) *Q.E.D.*

Question XXV.

"In a plane triangle ABC are given two sides, AC, BC, and the line CD = BC drawn from the vertex C to terminate in and bisect the base AB, to construct the triangle geometrically."

Answer. Let CA (fig. C.) = one of the given sides; with the other given side CB, describe BD; from A, draw the tangent AE, on which describe the semicircle AFE; with AF = FE describe FD, draw ADB and it is done. For $AF^2 + FE^2 = 2\,AF^2 = AE^2 = AD \times AB$; but AD = AF *per* construc. ∵ $AD \times AB = AF \times \overline{AF + DB}$ $= AF^2 + AF \times DB$; hence $2\,AF^2 = AF^2 + AF \times DB$, consequently AF (AD) = DB. *Q.E.D.*

Question XXVI.

"In lat. 53° N. stands a Tower the shade of whose summit on Tuesday *June* 9, 1772, described a curve on the plane of the horizon whose tranverse axis was 150 yards; required the height of the said Tower geometrically."

Answer. It is too well known to need demonstrating here, that when the sun's declination is less than the complement of latitude, the curve formed by the section of the horizontal plane with the Cone of Rays is an hyperbola. Therefore on the given line AB (fig. D.) = the transverse axis, describe the segment of a circle to contain an angle of twice the sun's given declination. Make the angle BAD = the difference of the colatitude and the said declination; and draw DC ⊥ AE, and it will be the height of the tower required. For let DE bisect

the angle BDF, and AG be perpend. to ED produced; then will the angle GDA = EDF = the codeclin. hence the ∠ GAD = the ſun's declination. And becauſe GE is evidently parallel to the earth's axis, the angle GAE = the colatitude; therefore the ∠ DAB = the difference of the colat. and the given declination. Moreover becauſe BDF is the cone of rays deſcribed by the ſun in his parallel, whoſe vertex is D, and AC the plane of the horizon cutting it in the hyperbola BH, AB is manifeſtly the tranſverſe axis; conſequently DC is the true height of the ſummit from the horizontal plane. The calculation is very eaſy; for in the triangle ADB, we have AB and all the angles to find another ſide, ſuppoſe AD; from whence, and the angle DAC, we find DC = 43.688.

N. B. I have taken the ſun's declination = 23°.

Queſtion XXVII.

" The ſhorteſt ſide of a right-angled triangle is given, and a perpendicular let fall from the right-angle cuts the hypothenuſe in extreme and mean proportion. It is required to conſtruct the triangle geometrically."

Anſwer. On the line AB (fig. E) drawn at pleaſure, make AD = the given ſide of the triangle; raiſe the indefinite ⊥ DC, and take BD to AD in the given proportion. Then make BC = DA, and join CA, and ABC will be the required triangle,

For BA is divided in the given ratio by the ⊥ DC, and BC = the given ſide *per* conſtruc. and ſince BA × BD ($=DA^2$) $= BC^2$, the ∠ at C is a right one, *per* Simpſon's Geom, *p*. 4. Theo. 19.

Queſtion

Queſtion XXVIII.

"In the common experiment of the double cone rolling (apparently) upwards, how far will it move between the indefinite right lines, which ſupport it, ſuppoſing they meet in an angle of 12 degrees, and the plane of ſituation elevated 4 degrees above the horizon, the common baſe of the double cone being 6 inches and diſtance between the vertices 13 inches?"

Anſwer. It is well known, in the experiment of the double cone moving apparently upwards, that the center of gravity thereof does actually deſcend; hence, ſhould the right lines or rulers which ſupport the cone be ſo poſited, as to cauſe the center of gravity (on its being rolled upwards) either to move horizontally, or to recede from the baſe or horizontal line, it manifeſtly can have no motion but what is cauſed by an external power.

It alſo appears from experiment, that the cone will move with a greater or leſs celerity, according as the right lines or rulers are inclined to one another in a greater or leſs angle, and that the angle may be made ſo acute, or the rulers brought ſo near together, as to cauſe the cone to roll downwards; conſequently there muſt be a certain angle, in which the cone will reſt in any part of the inclined plane —— now this angle is very eaſily determined as follows: Let ABCD (fig. F.) be an horizontal plane, and AIHD a plane inclined to it in the angle FMG. Let alſo MH, MI, repreſent the indefinite right lines or rulers, and KMLE a vertical ſection of the double cone.

Then becauſe FG is parallel and equal to NM, and HI parallel and equal to KL, it is evident, that while the plane KMLE

KMLE ſlides vertically betwixt the lines MH, MI, (which is the ſame thing as the cone rolling) the point N will move horizontally on the line NF, and when it arrives at F, the lines KL, HI, will coincide. And moreover, from this equality and parallelifm of the lines, we have this Analogy, as NM:NL, or as EM : KL :: FG : FI. But if we call FM radius, FG will be the ſine of the angle FMG, and FI the tangent of the angle FMI; therefore as the diameter of the common baſe of two cones, is to the diſtance of their vertices, ſo is the the ſine of the plane's elevation above the horizon, to the tangent of half the angle included by the two right lines or rulers. Hence, ſhould the given angle be leſs than that found by this proportion, it is plain, that the cone cannot aſcend, ſince on its being rolled upwards, the center of gravity muſt move upwards likewiſe. Let this now be applied to the preſent caſe, and we ſhall find that the angle, or inclination, of the indefinite right lines, ought to be ſomething more than 17° 12′, to cauſe the leaſt apparent aſcent of the cone; but this exceeds the given angle in the Queſtion (viz. 12°) conſequently the cone can in this caſe have no motion of itſelf. This will appear ſtill plainer, if we ſuppoſe the cone to be rolled along the inclined plane, to any given diſtance (ſuppoſe 10 inches) from the point of contact of the rulers; for its center of gravity in that ſituation would be found by calculation to have receded from the horizontal baſe line 6·212 inches, and conſequently the center of gravity would have aſcended ·212 of an inch, which is contrary to both theory and experiment.

We might purſue this matter ſtill farther, by ſhewing the greateſt angle of elevation of the inclined plane, as well as the greateſt or leaſt dimenſions of the cone, whereby it could poſſibly aſcend. But as all this muſt be very obvious to any one

one who duly attends to the principle of gravitation, it is needless to say any thing more.

Question XXIX.

Required a general rule for the inscribing of regular Polygons in a given circle.

The rule given by Mr. Malton in his Royal Road to Geometry, Prob. 25. which is also the same with that in Ward's Mathematician's Guide, Prob. 20. is this,

" Draw a diameter AB, on which construct an equilateral triangle ADB; or draw the two arks only, intersecting at D, (this preparation is the same for any polygon whatever;) then divide the diameter into as many equal parts as the polygon, required, has sides; and through the second division, from either extreme, draw a right line from D to the opposite side of the concave circumference." Upon this Prob. Mr. Malton makes the following remarks:

" Thus may a side of any polygon whatever, contained in a circle, be obtained; by observing the rule given above. And it is truly worthy of notice, that any right line drawn from D, cutting the diameter and the concave circumference, will cut them both in the same proportion; or in whatever ratio one of them is divided, a right line being drawn, from D, through the point of division, will also cut the other in the same ratio."

" Of this construction or equal division of the diameter and the circumference, no demonstration can be given, having consulted several able Geometricians concerning it; who say, that it is only an approximation, and not mathematically true. Yet

Yet I muſt own, that I do believe it to be perfectly true, or it could never anſwer ſo very accurately, as it does, in all diviſions whatever."

It ſeems to me ſomething very extraordinary to ſee a profeſſed Geometrician reaſon ſo very *ungeometrically.* I always thought that not even a mere reader, much leſs a reformer of Euclid, could give his aſſent to the truth of a geometrical conſtruction barely from a ſeeming concurrence of points, or coincidence of lines; but from an obvious regular deduction from firſt principles. For I am very clear, that there can be nothing effected by lines (at leaſt in plane Geometry) but a demonſtration may be given, directly or indirectly, of its truth or falſity. If Geometry were founded on no better a baſis than the bare teſtimony of external ſenſe, I am afraid we ſhould ſoon view the whole fabric in ruins. Mr. Malton, through his whole performance, ſeems to lay a great ſtreſs on an *ocular* demonſtration. From whence it ſhould ſeem, that in order to become a proficient in geometry, it is indiſpenſably neceſſary to be furniſhed with the whole apparatus of a good microſcope, which ſhould be the criterion of every linear conſtruction. It is to be hoped, however, that the following inveſtigation will fully convince this Gentleman (without relying wholly on our *optic* faculty) that this rule is ſo far from being "perfectly true" for all regular polygons, that it anſwers in one caſe only, when the coſine of the angle at the center ſubtended by the ſide of the polygon is equal to half the radius, which is eaſily ſhewn to be the property of an arch of 60 or 120 degrees, anſwering to the trigon or hexagon.

It is ſomewhat ſurprizing that ſo many able Mathematicians ſhould be conſulted, in order to be ſatisfied of the truth or falſity of this rule, which may be ſo eaſily demonſtrated in the following manner.

Biſect

Bisect AB (fig. G.) the diam. of the given circle in C, through which perpendicular to AB, draw indefinitely HD. Take AI to AB as AG is to AGB, and draw GID. Then must D evidently be the true point, from whence a right line being drawn through the given point I, will divide the diam. and concave circumference in the same ratio. Draw GF perpendicular to AB; then will CD be a fourth proportional to IF, IC, and FG. But FG is the sine, and FC equal the cosine of the arch AG, and CI is known from AB and AI being given; hence we have this analogy for finding the point D, so that DG shall divide AB and AHB in the same ratio. As the cosine of AG—CI : sine of AG :: CI : CD. Thus for the trigon, if we call AC, 1, AI will be equal $\frac{2}{3}$, and thence IC equal $\frac{1}{3}$ ∴ As Cos. 60°—$\frac{1}{3}$: S. 60° :: $\frac{1}{3}$: 1·73205 =CD. After the same manner we find the distance CD for the regular polygons as follow:

Pentagon, as Cos. 72°—$\frac{1}{5}$: S. 72° :: $\frac{1}{5}$: 1·74478, =CD.

Hexagon, as Cos. 60°—$\frac{1}{3}$: S. 60° :: $\frac{1}{3}$: 1·73205, =CD.

Heptagon, as Cos. 51°, 21′ $\frac{5}{7}$—$\frac{3}{7}$: S. 51°, 21′ $\frac{5}{7}$ :: $\frac{3}{7}$: 1·71903, = CD.

Octagon, as Cos. 45°—$\frac{1}{2}$: S. 45° :: $\frac{1}{2}$: 1·707106, =CD.

Nonagon, as Cos. 40°—$\frac{5}{9}$: S. 40° :: $\frac{5}{9}$: 1·69654, =CD.

Decagon, as Cos. 36°—$\frac{3}{5}$: S. 36° :: $\frac{3}{5}$: 1·68728, =CD.

Undecagon, as Cos. 32°, 43′ $\frac{7}{11}$ — $\frac{7}{11}$: S. 32°, 43′ $\frac{7}{11}$ :: $\frac{7}{11}$: 1·679165 = CD.

Duodecagon, as Cos. 30°—$\frac{2}{3}$: S. 30° :: $\frac{2}{3}$: 1·67202, =CD.

Now,

Now, by Malton's (Ward's) rule CD is equal $\sqrt{\overline{AD}^2-\overline{AC}^2}$ $= \sqrt{3} = 1{\cdot}73205$; which corresponds only with the trigon or hexagon. The reason why this construction answers to the arch of 60 or 120° will be evident if we consider, that in this case, CD is double of GF, and thence FI is $\frac{1}{3}$ of FC; but FC is $= \frac{1}{2}$, and therefore AI is $= \frac{1}{3}$ of AB; which is the same part as the arch AG is of AHB. Hence it plainly appears, that the point D found by Ward's method is not true for any polygon whatever, excepting in one single case; for it is evident that the side of the tetragon cannot be said to be found by this construction; for by the above analogy it will be, as Cos. 90°—1 : S. 90° :: 1 : $\frac{1}{0}$, which shews that CD in this case is infinite.

The latter part of Mr. Malton's 47th Prob. (which is also Ward's) is in the same predicament with the other, being proved to be false as follows:

The construction being made according to the rule (fig. H*i*) it is evident from the like situation of the circles, &c. that wherever the point B is taken in GK the lines EF, DC, will be always parallel to each other, and the ∠ DCF = CDE, as also DEF = CFE. The sides CF, FE, and ED, are likewise equal, being each a radius of the same or equal circles. What remains then to be proved is, whether the ∠ DCF be ∓ the greater angle in a regular pentagon, formed by the diagonal and a side. The most direct (if not the only) method of investigating which appears to be by a calculation of that angle.

Having drawn such lines as appear by the fig. EF will be the side of a regular hexagon, and FB the side of a regular dodecagon, inscribed in the same circle, and they are therefore in

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in the ratio of 1 to $\sqrt{2-\sqrt{3}}$. And ſince FG is $\perp$ IK, and KC $\parallel$ FG, the $\angle$ GKC is a right $\angle$; and the $\angle$ KBC$=$ABI is evidently $=\frac{1}{2}$ a right $\angle$. Alſo, as BF is the ſide of a dodecagon, the $\angle$ FIB $=\frac{1}{3}$ of a right $\angle$, and conſequently the $\angle$FBI $=\frac{5}{6}$ of a right $\angle$; but the $\angle$ CBF is the ſup. of KBC $+$ FBI $\therefore$ the $\angle$ FBC $=\frac{2}{3}$ of a right $\angle$ $=60°$. Then as the ſines of the $\angle$s FBC, FCB are in the ratio of their oppoſite ſides FC, FB, we have, as $1 : \sqrt{2-\sqrt{3}} ::$ S. 60° : S. 26°, 38′, 2″ nearly; hence the $\angle$DCF$=$DCB$+$BCF$=$71°, 38′, 2″. But the $\angle$ DCF in a regular pentagon is 72°. Therefore this conſtruction is falſe alſo.

N. B. Since the above was written, the inveſtigation of the veracity of theſe rules has been propoſed in the Ladies Diary, to which the above anſwers were ſent with ſome alterations.

Queſtion XXX.

To find the ſum of a ſeries conſiſting of 1000 surſolid numbers whoſe roots are $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, &c.

Anſwer. The general expreſſion for the ſum of a ſeries of powers as $2^n+3^n+4^n$, &c. to $\overline{r-1}$ terms is $\frac{r^{n+1}}{n+1}+\frac{r^n}{2}+\frac{nr^{n-1}}{3.4}$, &c. $-\frac{1}{n+1}-\frac{1}{2}-\frac{n}{3.4}+\frac{n+\overline{n-1}\times\overline{n-2}}{2.3.4.5.6.}$, &c. And ſince the roots of the propoſed ſeries are evidently in Arith. Progreſs. and the common difference equal to the firſt term, we may eaſily adapt this general equation to the given ſeries by adding unity to

to both sides thereof, and multiplying the whole by $\overline{\frac{1}{6}}\Big|^5$. For then it becomes $\overline{\frac{1}{6}}\Big|^5 + \overline{\frac{2}{6}}\Big|^5 + \overline{\frac{3}{6}}\Big|^5 \text{ - - - - } + \overline{\frac{r}{6}}\Big|^5 \Big(= \overline{\frac{1}{6}}\Big|^5 + \overline{\frac{1}{3}}\Big|^5 + \overline{\frac{1}{2}}\Big|^5 \text{ - - - - - } + \overline{\frac{r}{6}}\Big|^5$ the given series$\Big) = \overline{\frac{1}{6}}\Big|^5 \times : \frac{r^{n+1}}{n+1} + \frac{r^n}{2} + \frac{nr^{n-1}}{3.4} - \frac{n \times \overline{n-1} \times \overline{n-2} r^{n-3}}{2.3.4.5.6.}$, &c. $\frac{1}{n+1} + \frac{1}{2} - \frac{n}{3.4}$, &c. (till it terminates) $= \overline{\frac{1}{6}}\Big|^5 \times : \frac{\overline{1000}|^6}{6} + \frac{\overline{1000}|^5}{2} + \frac{5 \times \overline{1000}|^4}{12} - \frac{\overline{1000}|^2}{12}$ $= 21497824502732{\cdot}767$ the sum required.

Question XXXI.

In a geometrical progression beginning from unity, having the common ratio and number of terms given, the sum of all the changes in the series will be expressed by this general theorem $\frac{\overline{r^b - 1} \times 1.2.3 \text{ - - - - - } n}{br - b} \times : 1 + 10 + 100 + 1000$, &c. - - - - - to b terms; where each term must be distinctly considered, and estimated according to its variable local value in the series. Required the demonstration or investigation?

Question XXXII.

If A be any given number, P the places of figures, and C the number of changes, then will the sum of all those changes be represented by this series $\frac{AC}{P} + \frac{10AC}{P} + \frac{100AC}{P}$, &c. - - - - - continued to P terms. Required the investigation.

Question XXXIII.

Required to find a fraction such, that being taken from its reciprocal the remainder shall be a square.

This

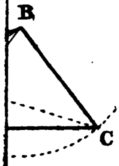
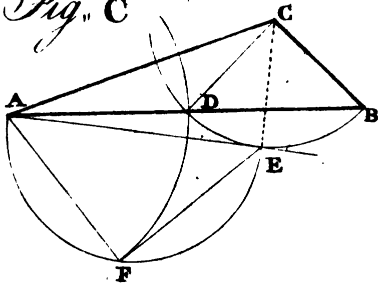


Fig. C



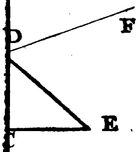
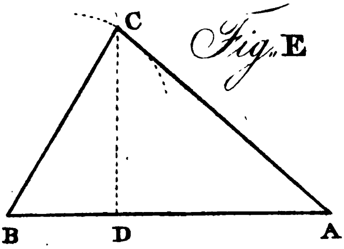


Fig. E



H

Fig. F.

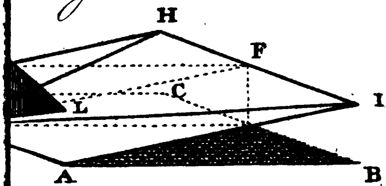
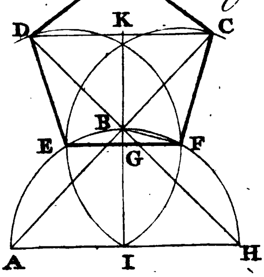


Fig. H

G

B



This question was proposed in the Ladies' Diary for 1768; but as the possibility, or impossibility, of such a fraction existing has never yet been shewn, and therefore the question not answered, I thought it not improper to do it here.

If we put $\frac{y}{x}$ for the required fraction; then, *per* question, $\frac{x}{y} - \frac{y}{x}$ is to be a square number; but $\frac{x}{y} - \frac{y}{x}$ is equal $\frac{x^2-y^2}{xy}$, therefore the question is, To find two numbers such, that their product and the difference of their squares may be square numbers. In order then to investigate two such numbers, if possible, put $xy = r^2$, then will $x = \frac{r^2}{y}$; therefore $\frac{r^4}{y^2} - y^2 = \square$ *per* quest. Now it is evident that $\frac{r^2}{y}$ will be in the same ratio to y, as the hyp. of a right-angled triangle is to a side; that is, as $h : s :: \frac{r^2}{y} : y$, $\because y = \sqrt{\frac{s}{h}} \times r$. From whence it appears that if a right-angled triangle can be found, of which the hyp. and either side are rational square numbers, the quest. may be answered, otherwise it is impossible. But that it is absolutely impossible for such a triangle to be formed, will be very evident from the general expressions for the sides of a commensurate right-angled triangle, *viz.* $m^2 + n^2$, $m^2 - n^2$, and $2mn$. For since $m^2 + n^2$, and $m^2 - n^2$ (or $2mn$) are both to be rational square numbers, it is plain that in the former expression m and n must themselves represent the two legs of a triangle; and in the latter m must also represent the hyp. and n a leg of a triangle; hence it follows, that either m must at the same time have two different values, or that there must be a right-angled triangle existing the hyp. of which is equal to one of its sides; both which conclusions are absurd.

The absurdity of this question may be also shewn rather differently, thus; Since $x^2 - y^2$ will be always a rational

ſquare number when $x = \frac{y^2+r^2}{2r}$, and xy will be always a rational ſquare when $x = \frac{r^2}{y}$, (where y and r may be taken at pleaſure) it appears, that if $\frac{x^2-y^2}{xy}$, be a ſquare number, $\frac{y^2+r^2}{2r}$ muſt neceſſarily be equal to $\frac{r^2}{y}$. Therefore if we ſuppoſe r of the ſame value in both expreſſions, (y being the ſame from the nature of the queſtion) we ſhall have $2r^3 = y^3 + yr^2$; and this ſuppoſition we have certainly a right to make, ſince the numbers are taken at pleaſure, conſequently if it can poſſibly be a ſquare number, it muſt evidently hold equally good in this caſe as in any other. Now in this equation nothing can be more obvious than if $r \sqsubset y$ that $2r^3$ will be greater than $y^3 + yr^2$, and if $r \sqsupset y$ that $2r^3$ will be leſs than $y^3 + yr^2$; therefore this equation cannot poſſibly obtain in rational numbers unleſs $r = y$, and this gives $x = y$ alſo; from whence nothing more can be inferred than that the required fraction muſt be $\frac{n}{n}$, or unity; which again ſhews the abſurdity of the thing. The queſtion therefore requires an abſolute impoſſibility.

It may, perhaps, be objected by the leſs diſcerning reader, that theſe are not ſufficient proofs of the abſurdity of this queſtion, as there are vulgar fractions which are equivalent to rational ſquare numbers, and yet neither numerator nor denominator independently ſuch; and therefore $\frac{x^2-y^2}{xy}$ may be a ſquare number, when neither x^2-y^2, nor xy are rational ſquares. To which it may be anſwered, that it can only ſo happen, when the numerator and denominator of a fraction are not prime to each other, or in their loweſt terms. But $\frac{x^2-y^2}{xy}$ muſt evidently be in its loweſt terms when $\frac{y}{x}$ is ſo; it is

is therefore impoffible that the expreffion $\frac{x}{y}-\frac{y}{x}$ can be a fquare number, unlefs both x^2-y^2 and xy are fuch, which has been before proved to be abfurd.

Another queftion equally as abfurd, which has been fome time handed about, is, To find a triangle fuch, that not only the fides may be whole numbers, but alfo a line drawn from the angle at the bafe, and terminating in the perpendicular, may be a whole number; and moreover, that the fquare of the leffer fegment, taken from the fquare of the whole perpendilar, may leave a whole fquare number.

Now the three fides of a triangle are only commenfurable when m^2+n^2 denotes the hypothenufe and m^2-n^2, and $2mn$ the two fides; m and n being numbers taken at pleafure, fo as $m \sqsubset n$. Let then AB (fig. 2.) be denoted by $2mn$, then can AD be expreffed only by m^2-n^2, if AB, AD, and BD muft be rational numbers. And fince AC^2-AD^2 is to be a whole fquare number, *per* queft. AD will be alfo the fide, and AC the hyp. of a right-angled triangle; but AD is equal m^2-n^2, and therefore AC can be reprefented only by m^2+n^2, to be a whole number; and fince AB is denoted by $2mn$, AC can be equal only to m^2-n^2. Hence it is evident, that if AB, AC, BC, BD, and AC^2-AD^2 muft be whole numbers, AC muft neceffarily be equal to both m^2+n^2, and m^2-n^2, which is abfurd.——But as this may not perhaps appear wholly fatisfactory to every reader, fince m and n may have different values, and yet $2mn$ which reprefents the bafe of the triangle continue the fame, I fhall confider it a little differently.——Let AB $=2mn$ (as before) $=2pq$, where $m \sqsubset n$, $p \sqsubset q$, $m \sqsubset p$, and $\therefore q \sqsubset n$, then will BC $=m^2+n^2$, AC $=m^2-n^2$, BD $=p^2+q^2$, and AD $=p^2-q^2$; but by the queft. AC^2+AD^2

$=\square$,

$= \Box$, that is $\overline{m^2 - n^2}|^2 + \overline{p^2 - q^2}|^2 = \Box$. Now it is obvious, since the sum of these two squares is to be a square whole number, that their roots must also represent the two sides of a commensurate right-angled triangle; but if $p^2 - q^2$ be one side thereof, the other can only be equal to $2pq$ when all the sides are rational, hence $m^2 - n^2 = 2pq = 2mn$, and AB $=$ AC, consequently BC ($= m^2 + n^2$) is irrational.

That AD and AC may be both expressed by $m^2 - n^2$, and BD and BC by $m^2 + n^2$, when the base is denoted by $2mn$, is evident, because, as we have just observed, the factors m and n may be variously assumed, and yet the base of the triangle retain the same numerical value. From whence it appears that there may be various right-angled triangles of rational sides, having one common base. To illustrate this, suppose we take the 18th Question of the 6th Book of Diophantus's Algebra, where it is proposed, that in the right-angled triangle ABC, AB, BC, CA, BD, AD, DC, shall be all whole numbers. Let $2mn =$ any compound number, so that m may be always greater than n; suppose 24; then will $m = 4$, or 6; and $n = 3$, or 2. From the first values of m and n, we have AD $= 7$, BD $= 25$, and AB $= 24$; and from the second values of m and n, AC $= 32$, BC $= 40$, and AB $= 24$; and thence DC $= 25$.——It may here be observed, that this is a general method of solving the problem; but Diophantus is obliged to draw the line BD, so as to bisect the angle ABC, in order to get an equation, *viz.* AB $\times$ DC $=$ BC $\times$ AD, to make the sides become rational.

If we take $2mn = 28$, and $m = \sqrt{28}$, or $\sqrt{98}$, and thence $n = \sqrt{7}$, or $\sqrt{2}$, (for $2 \times \sqrt{28} \times \sqrt{7} = 28$, and $2 \times \sqrt{98} \times \sqrt{2} = 28$) we shall have AD $= 21$, BD $= 35$, AC $= 96$, BC $= 100$, AB $= 28$, and thence DC $= 75$; which are the numbers found by Diophantus's method.

Hence it appears, that it is equally as easy to find a right-angled triangle, with any number of lines drawn from either of the acute angles, and terminating in the opposite side, so that not only the sides of the triangle, but also the lines so drawn, and the segments of the opposite side formed thereby, shall be all whole numbers; provided we can find a compound number to represent half the base, which may be resolved into as many different factors.

Question XXXIV.

* In triangulo plano ABD, angulus DAB obtusus est, angulus DAC rectus. Dantur AD + BD = 51, AC perpendic. ad AD = 21, AB = 32. Quæritur AD. (fig. 3.)

Solut. Ex A dimitte perpendiculum AE. Sit AB $= a$, AC $= b$, AB + BD $= c$, AD $= x$; tunc (ex natura triangulorum) $\frac{BD^2 + AD^2 - AB^2}{2BD}$ dat DE $= \frac{c^2 - 2cx + 2x^2 - a^2}{2c - 2x}$. Et ob similia triangula ACD, EAD; erit DC : DA :: DA : DE, et (*per* Euc. 47. 1.) DC $= \sqrt{AD^2 + AC^2}$, ergo $\frac{AD^2}{\overline{AD^2 + AC^2}|^{\frac{1}{2}}}$ $=$ DE. Exinde erit $\frac{c^2 - 2cx + 2x^2 - a^2}{2c - 2x} = \frac{x^2}{\sqrt{x^2 + b^2}}$, hinc æquatio numerosa $- 8072x^4 + 501636x^3 - 9856921x^2 + 141873228x = 1096735689$, ex hoc $x = 13{\cdot}6792$. *Q. E. D.*

* This question, with the solution, as they are here inserted, were sent to the Editor of the Town and Country Magazine in Jan. 1774, and the question was accordingly proposed in Feb. and answered in the next Magazine by Mr. G. B. of Coventry. But as the solution there given (if it may be called one) is very inelegant, and the conclusion false, I thought this a proper opportunity of giving the true one. It is evident, from a comparison of the solutions, that he has either committed an error in the reduction of his final equation, which would arise to very high dimensions; or else he has found the measure of AD from a mechanical construction, which is little better than guessing at the answer.

Question XXXV.

There are three ports ABE, whose bearings from each other are as follow, *viz*. B from A, *N* by *E* ¾ *E*; E from A, *E* by *S*, and E from B, *S. S. E*. A ship at the port A being bound to a certain island, C, bearing *N E*, saileth thence *E* by *S*, and after running as many miles (by the log.) as the port E is distant from A, in a current which setteth *N N E*, arrives at her desired haven, which is 100 miles distant from B. Quere the distance of the places from each other, and the velocity of the current, without Algebra? (fig. 4.)*

Answer. The figure being constructed as *per* question, In the trapezium ABCE we have BC = 100, and the ∠'s BAE, = 87°, 11′, 15″, AEB = 56°, 15′, ABE = 36°, 33′, CAE = 56°, 15′, AEC = 101°, 15′, and ACE = 22°, 30′; from whence we find BAC = 30°, 56′¼, and BEC = 45°. Now if the sines of all the angles be drawn to any determinate radius, it will readily appear from a similarity of triangles, and the composition of their ratios, that as the sin. BAC × sin. ACE : sin. ABE × sin. BEC :: BD : DC :: 1967 : 4202 :: 1 : 2·136,

* This question appeared in the Town and Country Magazine, for Sept. 1772, and was answered by the proposer Mr. C, in the next Magazine, the above solution being (as I was informed) sent too late for the month. What this Gentleman there gives us as a solution, may with just as much propriety be called so, as saying the drawing of a geometrical figure is the demonstrating of its properties. For it plainly appears from the question, that the method of investigating the numerical values of the distances and the veloc. of the current *without having recourse to Algebra*, was the thing required, and not a bare construction of it, which, by the bye, he has also rendered quite false, by taking the ang. NAB much less than that given in the question. The demonstration and calculation he has omitted, because, he says, they are too evident to be insisted on; but in this I cannot help thinking he is much mistaken, for I am pretty certain that no one can see the reason of the analogy, upon which the solution wholly depends, without some recollection.

The

The sum of the ang. at the base of the triang. BDC is 112°, 30′; therefore (*per* the third axiom of plane trigon.) it will be, as 3·136 : 1·136 :: tang. 56°, 15′, : tang. 28°, 28′, half the difference of those angles; whence the ∠ CBD is 84°, 43′, and BCD 27°, 47′. These being obtained, we easily find all the distances, that is, BA = 90·66, BE = 108·92, AE = 64·94, AC = 166·11, and EC = 140·82. And since EC is ∥ to Aa (the direction of the current) *per* construc. therefore as AE : EC :: 1 : 2·13, which is the ratio of the ship's velocity to the velocity of the current, which are all obtained without Algebra. *W. W. R.*

Question XXXVI.

"In a plane triangle, having given the vertical angle, the difference of the base and one side, and the sum of the perpendicular, from the angle of the base contiguous to that side upon the opposite side, and the segment thereby cut off from that opposite side contiguous to the other angle at the base; to construct the triangle*."

* This question was proposed in the Ladies' Diary for 1774, by the Rev. Mr. Lawson, to which two different solutions were inserted in the last year's Diary by the Rev. Mr. Wildbore, and the author of this Treatise. But as neither of these solutions were concise enough for the conductor of the mathematical correspondence in the Town and Country Magazine, he reproposed it; hoping that some of his ingenious contributors would send a more elegant solution. How far this has been performed I shall not pretend to determine; but the method of construction given in the Town and Country Magazine is certainly more simple than either of those in the Diary. But as the figure itself is quite false, owing to a want of parallelism in the lines, the triangle produced does not in any one respect correspond to the data; the vertical angle being much too large, and the assumed difference in the question being double of that in the triangle. Upon this account, and the want of a demonstration, which perhaps may not appear so evident (at least to the young reader) as he seems to think, I have inserted it here and corrected the figure.

CONSTRUCTION.

"Make AB (fig, A.) equal to the fum of the perpendicular and fegment, the angle ABC, 45°, and ABE the fupplement of the given vertical one, BE the given difference. Produce CB to meet FED drawn parallel to AB, in D. Join the points A, D, draw BH = BE; from A draw AC and CG parallel to BH and BE refpectively, then will AGC be the triangle required."

DEMONSTRATION,

Draw BK ‖ to DA; and join EH, and GK. Then, fince AB is equal to the given fum of the perpendicular and fegment, and the ∠ ABC half a right angle, by conftruction, a perpendicular let fall from any point as C upon the oppofite fide will be equal to *a*B (Eu. 9. 4.) ∴ A*a* + *a*C = AB, the given fum. And as the ∠ ABE is equal to the fup, of the given ∠ by conftruc. and CG ‖ to BE, the ∠ CGB is equal to the ∠ ABE (Eu, 29. 1,) and therefore the ∠ CGA equal to the given vertical angle, And moreover, becaufe DE, DH, DB, BH, BE, are refpectively ‖ to BG, BK, BC, CK, CG, the triangles BEH, CGK are fimilar. But BH is = BE ∴ CK is = CG. And again, becaufe DA is ‖ to BK, and KA ‖ to BH, KA is = BH = BE = the given difference of the bafe CA, and one fide CG. *Q. E. D.*

Queftion XXXVII.

In the mound of an Elliptical Garden, whofe tranfverfe is to the conj. as 3 to 2, a pedeftal, 50 yards high, is fo placed, as that the apparent magnitude of an Herculean Figure, 10 feet high, on the top of it, is the greateft poffible to an eye fituated on the tranfverfe, and 20 yards from the center of the Ellipfe. Required from hence the Area of the Garden.

Anfwer. In Fig. 5. EF reprefents the pedeftal ⊥ to the plane of the femi-ellipfe ABI; FG the ftatue, and D the given point in the tranfverfe. Now it is evident fince DE is in

the

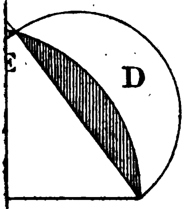
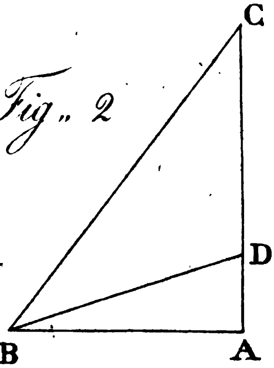


Fig. 2



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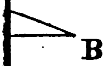


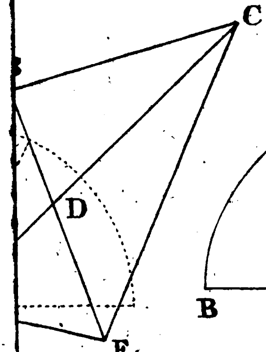
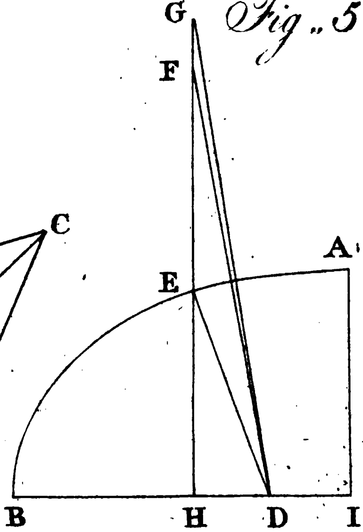


Fig. 5



the plane of the ellipse, and GE ⊥ to that plane, that the ∠ GED is a right one, and therefore if we call GE rad. ED will be the tang. of the ∠ G, and $\frac{ED \times GE}{FE}$ that of the ∠ F. Moreover it is evident, that the ∠ FDG under which the object is seen, is the difference of the angles DFE, DGE, which must be a max. *per* question. Making therefore FE $= a$, EG $= b$, and ED $= t$, we have $\frac{bt}{a} =$ the tang. of the greater angle, and $t =$ that of the less; and the flux. of the difference of these angles, expressed in terms of the rad. and tang. is $\frac{b^2\dot{t}}{b^2 + t^2} - \frac{ab^3\dot{t}}{ab^2 + b^2t^2}$, which being made $= 0$, and reduced, gives $t = \sqrt{ab}$. Hence it appears that DE the dist. of the pedestal is a mean proportional between GE and FE.

Now, it is obvious, from the nature of the quest. that the line DE must be the nearest dist. to the curve, or a min. therefore, putting ID $= c$, and the sem. transv. $= x$, we have (*per* quest. and *Emerson's* Flux. *p.* 126) $\frac{4c}{5} =$ DH; hence (Euc. 47. 1.) $ab - \frac{16c^2}{25} = \text{HE}^2$, and *per* prop. of the Ellipse, $\frac{4}{9} \times \overline{x^2 - \frac{81c^2}{25}} = \text{HE}^2$; $\therefore \frac{4}{9} \times \overline{x^2 - \frac{81c^2}{25}} = ab - \frac{16c^2}{25}$; reduced $x = 245{\cdot}9385$, and thence the area of the Ellipse $= 126681{\cdot}1$ square feet *.

Question

* This question I proposed in the Ladies' Diary for 1775, to which a solution is given in the Diary for the year following by Mr. Rowe; but, by a mistake in the conclusion of it, he has brought out the required area of the ellipse only about one half what it ought to be.

The same question was re-proposed by the conductor of the mathematical department in the T. and C. Mag. for Aug. 1776, occasioned chiefly, I imagine, by the mistake in Mr. Rowe's solution.——And here, in order to do justice to the author of the Diary and myself, I am under the disagreeable necessity of informing the reader, that the note subjoined to the solution

Question XXXVIII.

From the fluxionary expression $\sqrt{16y^4 \cdot \overline{a^2 - y^2} \cdot \dot{x}^4 \dot{y}^4} - 4d\dot{x}\dot{y}^3 = 0$. Required a finite value of x in terms of y and known quantities.

Answer. The expression by transposition, &c. becomes $4\dot{x}^2\dot{y}^2y^2\sqrt{a^2 - y^2} = 4d\dot{x}\dot{y}^3$. Divide by $4\dot{x}\dot{y}^2$, and there arises $\dot{x}y^2\sqrt{a^2 - y^2} = d\dot{y}$; hence $\dot{x} = \frac{d\dot{y}}{y^2\sqrt{a^2 - y^2}}$. Now by

solution of this question in the Mag. for the month following, contains assertions that I never wrote or even once thought of; which the following letter will evince, being copied verbatim from that I sent with the above solution to the Editor of the T. and C. Magazine.

"Sir,

"As it will be naturally expected that I should offer a solution to my "own question, I have here sent one nearly the same with that I gave to "the author of the Diary.—The restriction you speak of is certainly "right, but I thought it rather unnecessary to mention it, because it is "obvious to every one that either this limitation must be understood, or "the situation of the pedestal in the periphery given; otherwise the question "would be absolutely unanswerable.—Mr. Rowe's solution is on the "same principle as the above; but is rendered false, I observe, by a mis-"take in the conclusion of it, having put down $3 \times \overline{.7854 \times ab + \frac{1}{2}cc}$ "instead of $6 \times \overline{.7854 \times ab + \frac{1}{2}cc}$, this latter expression bringing out the "same numbers as the above.

"I am, &c."

I remember Terence somewhere says, *Veritas odium parit*—which perhaps may be the case here; but notwithstanding this, I cannot help observing, that whoever compares this letter with the note before-mentioned will hardly forbear tearing it out of the Mag. as a downright falsehood. —It is a pity that this entertaining and curious miscellany should be prostituted to such low purposes in the Mathematical parts, as to be a vehicle for malevolence and spleen; the reproposing of this quest. in the Mag. being evidently only another weak effort to depreciate (if possible) the mathematical character of the present author of the Ladies' Diary; for which purpose, it seems, truth or falsehood is equally subservient.

Emerson's

Emerson's Tables, Forms 10th and 12th, we get the fluent of $\frac{\dot{y}}{y^2\sqrt{a^2-y^2}} = -\frac{\sqrt{a^2-y^2}}{aay}$; and therefore the fluent of $\frac{d\dot{y}}{y^2\sqrt{a^2-y^2}} = \frac{-d\sqrt{a^2-y^2}}{a^2y}$, whence $x = -\frac{d\sqrt{a^2-y^2}}{a^2y}$; the Neg. sign shewing that it would require a correction according to the nature of the problem.

Question XXXIX.

In the expression $\frac{zy\dot{y}^3}{\dot{z}^2}$, required the relation of y and z, supposing their nascent increments to be cotemporaneous, and the fluent corresponding to any given values thereof a minimum; also the curve defined by the equation expressing the relation of y and z.

Answer. Let $\dot{y}$ alone be considered as variable, then the Flux. will be $\frac{3zy\ddot{y}\dot{y}^2}{\dot{z}^2}$; and when y only is made variable, its Flux. becomes $\frac{z\dot{y}^4}{\dot{z}^2}$. Let now the latter be divided by the former, and the quotient put $= \frac{t}{\dot{t}}$, and we have $\frac{\frac{1}{3}\dot{y}}{y} = \frac{\dot{t}}{t}$: Hence the Hyp. Log. t = Hyp. Log. $\sqrt[3]{y}$ + Hyp. Log. m, (m being any constant quantity), therefore $t = my^{\frac{1}{3}}$; which being equated with the Flux. of the given expression, when $\dot{y}$ alone is variable, gives $\frac{3zy\ddot{y}\dot{y}^2}{\dot{z}^2} = my^{\frac{1}{3}}$, from whence $y^{\frac{2}{3}}\dot{y}^2 = \frac{1}{3}mz^{-1}\dot{z}^2$, the Fluent of which is $\frac{3}{4}y^{\frac{4}{3}} = 2\sqrt{\frac{1}{3}mz}$, expressing the relation of y and z. If for $\frac{8}{3}\sqrt{\frac{m}{3}}\Big|^{\frac{3}{8}}$ we put d, then the equation becomes $y^8 = d^3z^3$ answering to a Parabola of the higher order.

Question

Question XL.

Required the lat. of the place, and declination of the sun, when the length of the day is to that of the night, in the ratio of 3 to 2, and the sun's meridian altitude to his depression at midnight, as 2 to 1. See T. and C. Mag. *p.* 303. 1773.

Answer. In the Orthographic scheme (fig. 6.) P represents the North Pole, EQ the Equator, HO the Horizon, AD the Sun's Semidiurnal Arc, and DR the Seminocturnal. From the given ratio of the lengths of the day and night, the arches AD, DR, are both known; therefore putting a = the nat. versed sine of AD, b = the nat. v. s. of DR, and x = (Rb) the nat. sine of ☉ R the Sun's depression at midnight, *per sim.* △'s $b : a :: x : \frac{ax}{b} = Aa$, the sine of the Sun's merid. altitude. But *per* quest. AH = 2 OR, that is $2x\sqrt{1-x^2} = Aa$, $\therefore \frac{ax}{b} = 2x\sqrt{1-x^2}$; hence $x = \sqrt{1-\frac{a^2}{4b^2}} = \cdot320604$ the nat. sine of 18°, 42′, the Sun's depression at midnight; ∴ by the nat. of the sphere the comp. of the req. lat. HE $(= \frac{1}{2} \times \overline{HA + OR}) = 28°, 3′$, and the Sun's declin. = EA $(= \frac{1}{2} \times \overline{HA - OR}) = 9°, 21′$. *W. W. R.*

Question XLI.

The Fraction $\frac{2\dot{1}3\cdot47\dot{6}}{4\dot{1}7\cdot\dot{6}}$ is equal to 511176 $\frac{314\frac{36116}{99999}}{417\cdot2}$. Required the investigation.

Question XLII.

If any plane triangle ABC (fig. 7.) be circumscribed by a circle, and a right line be drawn from any one of the angular points, suppose B, bisecting the said ang. till it meets the circumference in D: I say that a circle descibed with rad. DA,

DA, will paſs through the center of a circle inſcribed in that triangle, and alſo through the other angular point C. Required a Geometrical demonſtration.

Queſtion XLIII.

To find the center of a circle P (fig. 8.) to paſs through a given point, and to cut two lines given in poſition, ſo that the intercepted arch AB may be of a given magnitude; or, that its chord may ſubtend a given ∠ APB.

Queſtion XLIV.

From a given point C (fig. 9.) on the diam. AB of a circle produced, let a tang. CD be drawn, and another right line from the point C, cutting the periphery in E and F, ſo that the ſine of the ∠ BCE, may be a fourth proportional to CB, (conſidered as radius) and the ſines of BCD and a given ∠ P. I ſay that the acute ∠ AIF or BIE, formed by the diagonals AE, BF, is equal to the given ∠ P. Required a Geometrical demonſtration.

Queſtion XLV.

Suppoſe a ſemicircular bowl placed on an horizontal plane, at what height and diſtance from the bowl muſt a light be placed, ſo as to illuminate one half the interior ſurface thereof?

Queſtion XLVI.

By repeated obſervations on the Northernmoſt ſtar in the right foot of Urſa Major (marked by Bayer ι) whoſe preſent declination is 49°, 15′, I have found that its altitude encreaſes more in a given time, in a certain latitude, than any other ſtar of different declination. Required the latitude of the place of obſervation, and the increaſe of the altitude of the ſaid ſtar ι from 8 to 11 when it ſouths at midnight.

Queſtion XLVII.

In the latitude 53°, 27′, N. on June 25, 1775, I ſaw a rainbow bearing *ESE*. Required the hour of the day.

Queſtion XLVIII.

Suppoſe an inflexible rod of iron, 40 feet long and 2 inches diameter, be ſo fixed at one end that it may vibrate freely. In what time will the other end of the rod, being let fall from an horizontal direction, deſcribe an arch of a given length, ſuppoſe 35 feet, from the commencement of motion?

Queſtion XLIX.

Required general expreſſions for the ſides of a right-angled triangle in whole numbers, ſo that any given number of lines drawn from one of the acute angles and terminating in the oppoſite ſide, as alſo the ſegments of the ſaid ſide formed thereby, may be all whole numbers. And moreover it is required, that the given number of lines ſo drawn may be the moſt which that triangle can poſſibly admit of in whole numbers; and that the periphery of the triangle may be a minimum. Suppoſe 5 the number of lines to be drawn, required the ſides of the triangle, &c. as above.

Queſtion L.

A cone being cut by a plane parallel to the baſe, the area of the ſection (= 20·7736 inches) is found to be a mean proportional between the ſuperficies of the two parts of the cone; and the ratio of the ſide of the lower fruſtum to the ſemidiameter of the baſe as 23 : 80. Required the dimenſions of the cone.

Queſtion

Question LI.

The expression $\sqrt{ax^3 - b}$, is a rational number. Required a rational value of x.

Question LII.

The internal diameter and diagonal of a cylindrical cask (which is made of the least wood possible) are respectively expressed by x^y, and $\overline{\overline{x^y}^{x} \times x}^{\frac{1}{2}}$. Required the content in Ale Gallons when $y = x^x$.

Question LIII.

Let $y - a = \overline{a^2 \times \overline{a - x}^{2}}^{\frac{3}{4}}$ be an equation to a curve. Quere the Abscissa when y is a maximum.

Question LIV.

How many elementary sounds may be formed out of the 24 letters of the alphabet * ?

Question LV.

It is required to divide the area of a circle geometrically into a given number of parts, which may be equal both in area and circumference †.

Some useful Remarks upon Equations.

As the young Algebraist generally meets with some difficulty in rightly ascertaining the roots of quadratic equations of the

* Tacquet in his *Arithmeticæ Theor.* p. 381, says. Mille milliones scriptorum mille annorum millionibus non scribent omnes 24 litterarum alphabeti permutationes, licet singuli quotidie absolverent 40 paginas, quarum unaquæque contineret diversos ordines litterarum 24.

† This *Paradoxical Question*, or perhaps rather *double entendre*, is taken from *Lawson's Dissertation on the Geometrical Analysis of the Antients.*——The demonstrations of all the theorems, with the Geometrical Constructions of the problems contained in this book, will be given at the latter end of *An Essay on the usefulness of Mathematical Learning*, which will soon be published.

3[d] form,

3^d form, as also in Cardan's and Colson's Formulæ for the roots of Cubics and Biquadratics, from the irrational Binomial $\pm x \pm \sqrt{-y}$. I suppose the following remarks will not be unacceptable to him.

1. In the solution of a problem where the final equation is in this form $x^2 - ax = -b$, and thence $x = \frac{a}{2} \pm \sqrt{\frac{a}{4} - b}$, the conditions of the problem must determine whether the affirmative or negative sign gives the true value of x; for if from the nature of the question, x be greater than a, we must evidently use the affirmative sign; and if less, the negative. As in this problem. *To divide the number* 100 (a) *into two such parts that their product and the difference of their squares may be equal to each other.* If we denote the lesser part by x, and therefore the greater by $a - x$, we shall find $x = \frac{3a}{2} \pm \sqrt{\frac{5aa}{4}}$. But x being given less than a, the upper sign (+) gives x too great; so that $x = \frac{3a}{2} - \sqrt{\frac{5aa}{4}} = 38{\cdot}19658$, &c. must be the true value required.

2. Hence it appears that though there be two affirmative roots in a quad. equation, yet, in general, only one of them will answer one case, or the particular question proposed. The same observation holds good in equations of all dimensions; for suppose in the solution of a question we have derived this final biquadratic equation, $x^4 - ax^3 + bx^2 - cx + d = 0$, where all the roots are affirmative, we must not conclude that the question admits of four different answers, for it will often be found upon trial, that three of the roots will produce an absurdity, and only one value answer the particular case proposed.

3. In

3. In a question producing a quad. equation of the third form, if the unknown quantity be assumed indeterminately in regard to *greater* and *less*, then will the affirmative and negative signs exhibit those values respectively. For if it was required to find two numbers whose sum is a and the sum of their alternate quotients b, we shall find that $\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \frac{a^2}{2+b}}$ are the numbers required.

4. In the reduction of cubic equations it will be proper to inform the young reader, that Cardan's rule is only of use in cases where two of the three roots are impossible; and therefore it would be in vain to attempt to solve a cubic equation composed of real roots by this method. As for example. Let the equation $z^3 - 7z = 6$ be proposed, whose roots are -2, 3, and -1; the numeral coefficients being written in the formula, we have $z = \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{72}}}$; which is only an imaginary expression; the square root of a negative quantity being impossible. So in this equation $x^3 - 91x = -330$, the roots of which are 5, 6, and -11, we get $x = \sqrt[3]{-165 + \sqrt{-685{\cdot}03}} + \sqrt[3]{-165 - \sqrt{-685{\cdot}03}}$, which is also imaginary. But when two of the roots are impossible as in this equation $x^3 + 6x = 20$; then we get $x = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = 2$, where the expression is real and possible; the other values of x being imaginary.

5. I shall now endeavour to clear up to the learner some seeming difficulties in finding the roots of a cubic equation by Colson's Theorem. And as this method principally depends

upon finding the cubic root of an impossible binomial, I shall first shew the investigation of an easy rule for obtaining such roots.

Let $a + \sqrt{b} = \sqrt[3]{A + B}$; then by involution $A + B = a^3 + 3a^2\sqrt{b} + 3ab + b^{\frac{3}{2}}$. Put $a^3 + 3ab = A$, and $3a^2\sqrt{b} + b^{\frac{3}{2}} = B$; then will $\overline{a^3 + 3ab}|^2 - \overline{3a^2\sqrt{b} + b^{\frac{3}{2}}}|^2 = a^6 - 3a^4b + 3a^2b^2 - b^3 = \overline{a^2 - b}|^3 = A^2 - B^2$, $\therefore b = \overline{aa - \overline{A^2 - B^2}|^{\frac{1}{3}}}$; hence, by substitution, $a^3 + 3a \times \overline{a^2 - \overline{A^2 - B^2}|^{\frac{1}{3}}} = A$, that is $4a^3 - 3\sqrt[3]{A^2 - B^2} \times a = A$; from whence a is easily found, and since $\sqrt{b} = \sqrt{a^2 - \sqrt[3]{A^2 - B^2}}$, we have $a + \sqrt{a^2 - \sqrt[3]{A^2 - B^2}} = \sqrt[3]{A + B}$.

When $\sqrt[3]{A^2 - B^2}$ is a surd, both members of the root will be irrational. In that case multiply the given equation by some number till $\sqrt[3]{A^2 - B^2}$ comes out rational, remembering to divide the values of a and $\sqrt{b}$ at the last by the root of that number.

6. Now if the second term of any cubic equation, reduced to Colson's general form, be exterminated, we shall have by Cardan's rule this value of the new root,

$$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}.$$

And as the cubic root of a binomial may be always investigated in a similar form, they assume $m \pm \sqrt{n} = \sqrt[3]{r \pm \sqrt{r^2 - q^3}}$, and thereby obtain $z = m + \sqrt{n} + m - \sqrt{n} = 2m$; but as this gives but one value of z, they derive two more expressions from this most obvious principle, *viz.* that the cube root of any number, being multiplied by the cube root of unity, must

must still remain a root of that number, that is, $\sqrt[3]{x^3} \times \sqrt[3]{1} = x$; but the three roots of unity are 1, $\frac{-1+\sqrt{-3}}{2}$, and $\frac{-1-\sqrt{-3}}{2}$: so it appears that the cube root of each binomial may be expressed in three different forms as follow:

$$\sqrt[3]{r + \sqrt{r^2 - q^3}} =$$

I. $\overline{m + \sqrt{n}} \times 1 = m + \sqrt{n}$.

II. $\overline{m+\sqrt{n}} \times \frac{-1+\sqrt{-3}}{2} = \frac{-m-\sqrt{n}+m\sqrt{-3}+\sqrt{-3n}}{2}$.

III. $\overline{m+\sqrt{n}} \times \frac{-1-\sqrt{-3}}{2} = \frac{-m-\sqrt{n}-m\sqrt{-3}-\sqrt{-3n}}{2}$.

$$\sqrt[3]{r - \sqrt{r^2 - q^3}} =$$

1. $\overline{m - \sqrt{n}} \times 1 = m - \sqrt{n}$.

2. $\overline{m-\sqrt{n}} \times \frac{-1+\sqrt{-3}}{2} = \frac{-m+\sqrt{n}+m\sqrt{-3}-\sqrt{-3n}}{2}$.

3. $\overline{m-\sqrt{n}} \times \frac{-1-\sqrt{-3}}{2} = \frac{-m+\sqrt{n}-m\sqrt{-3}+\sqrt{-3n}}{2}$.

From whence we shall evidently have these nine different expressions for z,

I. + 1. $= 2m$ - - - - - - - - - - - $= z$.

I. + 2. $= \frac{m+3\sqrt{n}+m\sqrt{-3}-\sqrt{-3n}}{2}$ - - $= z$.

I. + 3. $= \frac{m+3\sqrt{n}-m\sqrt{-3}+\sqrt{-3n}}{2}$ - - $= z$.

II. + 1. $= \frac{m-3\sqrt{n}+m\sqrt{-3}+\sqrt{-3n}}{2}$ - - $= z$.

II. + 2. $= m + m\sqrt{-3}$ - - - - - - - $= z$.

II. + 3. $= -m + \sqrt{-3n}$ - - - - - - - $= z$.

 III.

$$\text{III.} + 1. = \frac{m - 3\sqrt{n} - m\sqrt{-3} - \sqrt{-3n}}{2} \quad - \quad = z.$$

$$\text{III.} + 2. = -m - \sqrt{-3n} \quad - \; - \; - \; - \; - \quad = z.$$

$$\text{III.} + 3. = -m - m\sqrt{-3} \quad - \; - \; - \; - \; - \quad = z.$$

But as the 6th and 8th expreſſions only correſpond in the form to the cubic root of the irrational binomial, they are therefore the moſt convenient for uſe.

7. Having compared the given equation with the general one and thence obtained the numerical values of q and r, it will then appear, whether $r^2 - q^3$ be affirmative or negative. And ſince it is evident from the rule, that whatever ſign the value of $r^2 - q^3$ is affected with in the binomial, the ſame ſign will the correſponding term (n) have in the root, it plainly follows, that if n (or $r^2 - q^3$) be negative the expreſſions $-m + \sqrt{-3n}$, $-m - \sqrt{-3n}$ will become real and poſſible, that is $-m + \sqrt{3n}$, $-m - \sqrt{3n}$, which no one can be ſo dull as not to ſee the reaſon of, if he knows that a negative quantity ſubtracted becomes an affirmative one. On the contrary if n be affirmative, $\sqrt{-3n}$ is abſolutely impoſſible, there being no ſuch thing (according to the common definition of the term) as the ſquare root of a negative quantity; thoſe two roots are therefore then imaginary, and $2m$ is the only poſſible value of z.

8. It appears from the laſt Rem. that Cardan's Theorem may be rendered generally uſeful, by ſolving ſuch equations as have three roots real, as well as thoſe that have but one root real and two impoſſible ones. For ſince the three values of z may be expreſſed by $2m$, $-m + \sqrt{-3n}$, and $-m - \sqrt{-3n}$, it follows, that whenever the expreſſion for the root becomes imaginary, by ſubſtituting the numerical values in the formula, that is when $r^2 - q^3$ is negative, it will

will be poſſitive in the formulæ for the cubic roots; and therefore Cardan's may be as conveniently uſed as Colſon's form.

Example 1.

Let $x^3 - 12x^2 + 41x = 42$ be propoſed. The ſecond term being exterminated, we have $z^3 - 7z = 6$; this, by ſubſtitution in the formula, becomes

$$z = \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}.$$

Here then by Rem. 5. we have $A = 3$, $B = \sqrt{-\frac{100}{27}}$, and $\sqrt[3]{A^2 - B^2} = \sqrt[3]{\frac{343}{27}} = \frac{7}{3}$, $\because 4a^3 - 7a = 3$; hence $a = -1 = m$, and $\sqrt{a^2 - \sqrt[3]{A^2 - B^2}} = \sqrt{1 - \frac{7}{3}} = \sqrt{-\frac{4}{3}} = \sqrt{n}$; theſe values being written in the formulæ for z give

$$z = 2m \quad - \quad - \quad - \quad - \quad - \quad = -2.$$
$$z = -m + \sqrt{-3n} = +1 + \sqrt{4} = +3.$$
$$z = -m - \sqrt{-3n} = +1 - \sqrt{4} = -1,$$

From whence the roots of the propoſed equation are,

$$x = -2 + 4 = 2.$$
$$x = +3 + 4 = 7.$$
$$x = -1 + 4 = 3.$$

Example 2.

Let the propoſed equation be $x^3 - 21x = -20$.

Here $x = \sqrt[3]{-10 + \sqrt{-243}} + \sqrt[3]{-10 - \sqrt{-243}}$.
From whence $\sqrt[3]{A^2 - B^2} = 7$; therefore $4a^3 - 21a = -10$, and the root $a = 2$, and $\sqrt{n} = \sqrt{-3}$; hence the three roots are,

$$x = 2m \text{ - - - - - - - - - - - } = +4,$$
$$x = -m + \sqrt{-3n} = -2 + \sqrt{9} = +1,$$
$$x = -m - \sqrt{-3n} = -2 - \sqrt{9} = -5.$$

Example 3.

Given $x^3 - 15x = 4$, required x.

The numbers being substituted in the formula, and the cubic roots extracted as before, we get, $2 + \sqrt{-1} = m + \sqrt{n}$; from whence the roots are found to be,

$$x = 2m = \text{ - - - - - - - - - - } 4,$$
$$x = -m + \sqrt{-3n} = \text{ - - } -2 + \sqrt{3},$$
$$x = -m - \sqrt{-3n} = \text{ - - } -2 - \sqrt{3}.$$

9. One root of any cubic equation may be had by Cardan's Form though the expression becomes impossible, without having recourse to the formulæ $m \pm \sqrt{n}$, by extracting the roots of the binomials, the impossible terms vanishing in the addition by being always affected with contrary signs.

As for example in this equation, $y^3 - 7y = 6$; here $y = \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}$ which is an impossible expression; but the cubic roots being extracted we get $-1 + \sqrt{-\frac{4}{3}} - 1 - \sqrt{-\frac{4}{3}} = -2$, a root.

So in this equation $y^3 - 15y = 4$, we have $y = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$; which is also imaginary; but the cubic roots being extracted, we get $2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$, a root.

Suppose it is required to exhibit, by Cardan's Rule, the three roots of this equation $x^3 + \sqrt[3]{54} \times x = 44$, where two of the roots are impossible and $\sqrt[3]{A^2 - B^2}$ irrational.

By

By the rule we find $x = \sqrt[3]{22 + \sqrt{486}} + \sqrt[3]{22 - \sqrt{486}}$; and by Rem. 5. $\sqrt[3]{A^2 - B^2} = -\sqrt[3]{2}$, a surd. Multiply by 2, and the value of x becomes $\sqrt[3]{\frac{44 + \sqrt{1944}}{2}} + \sqrt[3]{\frac{44 - \sqrt{1944}}{2}}$; hence $\sqrt[3]{A^2 - B^2} = -\sqrt[3]{8} = -2$, $\because 4a^3 + 6a = 44$, and the root is $a = 2$; so $\sqrt{b} = \sqrt{6}$, and $2 + \sqrt{6} + 2 - \sqrt{6} = 4$. Therefore, $\frac{4}{\sqrt[3]{2}} = 2^{\frac{5}{3}}$ is the real root of the proposed equation, and the two impossible roots are $\frac{-2+\sqrt{-18}}{\sqrt[3]{2}}$, and $\frac{-2-\sqrt{-18}}{\sqrt[3]{2}}$. For $\overline{x + \frac{2 - \sqrt{-18}}{\sqrt[3]{2}}} \times \overline{x + \frac{2 + \sqrt{-18}}{\sqrt[3]{2}}} \times \overline{x - 2^{\frac{5}{3}}} = x^3 + \sqrt[3]{54} \times x = 44$, the proposed equation.

10. As the solutions of biquadratics depend upon the cubic or quad. equations, there can no more difficulties occur therein than those we have already explained. In regard to general formulæ for the sursolid and higher equations, I think they may, as yet, remain among the Mathematicians *desiderata*, as there does not appear to be any method of obtaining such finite expressions, without some particular relation of the roots, or of some of the coefficients. I shall give an example or two of this kind, where there is a particular relation of the coefficients to the absolute number; or of one coefficient to the rest.

Example 1.

Let $x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$, and let $E = \frac{DA}{2} - \frac{CA^2}{4} + \frac{BA^3}{8} - \frac{A^5}{32}$; then will the roots of the following equations be those of the proposed one, *viz.*

$$x^4 + px^3 + qx^2 + rx + s = 0, \text{ and } x + p = 0.$$

Where $p = \frac{A}{2}$, $q = B - \frac{A^2}{4}$, $r = C - \frac{AB}{2} + \frac{A^3}{8}$, and $s = D - \frac{CA}{2} + \frac{A^2B}{4} - \frac{A^4}{16}$. For the two equations multiplied produce

$$\left.\begin{matrix}x^5 + p \\ + p\end{matrix}\right\} \times x^4 \left.\begin{matrix}+ q \\ + pp\end{matrix}\right\} \times x^3 \left.\begin{matrix}+ r \\ + pq\end{matrix}\right\} \times x^2 \left.\begin{matrix}+ s \\ + pr\end{matrix}\right\} \times x + ps = 0.$$

And the coefficients being equated, we get the values of p, q, r, s, and the particular relation of E as above.

Let this formula be applied to the numeral surſolid equation $x^5 + 26x^4 + 225x^3 + 820x^2 + 1244x + E = 0$, where the coefficients are aſſumed at pleaſure. Here then $A = 26$, $B = 225$, $C = 820$, $D = 1244$, and therefore $E\left(= \frac{DA}{2} - \frac{CA^2}{4} + \frac{BA^3}{8} - \frac{A^5}{32}\right) = 624$, the abſolute number; hence $p = 13$, $q = 56$, $r = 92$, and $s = 48$; then will the roots of the biquadratic $x^4 + 13x^3 + 56x^2 + 92x + 48$, be found to be -1, -2, -4, and -6, and conſequently thoſe of the propoſed equation will be -1, -2, -4, -6, and -13.

Example 2.

Suppoſe the ſecond term exterminated, and let the reſulting equation be $z^5 + Bz^3 \pm Cz^2 \pm Dz + E = 0$, and let $D = BR^2 - \frac{RE}{B}$, R being the root of this equation $a^3 - Ba = \frac{E}{B} - C$; then will the five values of z be expreſſed by the roots of theſe two equations $z^3 + az^2 + a^2z + \frac{c}{d} = 0$, $z^2 - az + d = 0$. For theſe equations being multiplied produce

produce an equation similar to that proposed, and the value of a is known from the above cubic equation; and by equating the coefficients we have also the values of b and c.

When this formula is applied to any numerical sursolid equation; as $z^5 + 16z^3 - 820z^2 + Dz + 320 = 0$, (the coefficients being assumed at pleasure) we first find the value of $a = 10$, and thence we get $D = 1440$, $d = 16$, and $\frac{c}{d} = 20$. Therefore the roots of this equation

$z^5 + 16z^3 - 820z^2 + 1440z + 320$, will be had by resolving the cubic, $z^3 + 10z^2 + 100z + 20 = 0$, and the quadratic $z^2 - 10z + 16 = 0$, the roots of which are, $+ 9{\cdot}93$, $+ 8$, and $+ 2$, the other two roots being imaginary.

11. Various other literal expressions may be found, by taking away some intermediate term, and then finding two equations, the product of which shall give a result similar to the proposed equation; but these will be found to be of use only in particular cases, which seldom occur in practice. For though there may be always as many independent equations as unknown quantities, yet it will be found, that by their different combinations, we shall always recur to an equation of the same dimensions with that of which we are endeavouring to investigate the roots. And if we strike out any one term in either of the assumed generating equations, we shall then have more equations than unknown quantities, from whence necessarily arises a particular relation between some of the coefficients.

12. But supposing a compleat formula for the roots of a sursolid, or an equation of higher dimensions, were by any algebraic artifice obtained, it would be of little value, for there would be far more trouble in getting the roots this way,

than by the method of converging series, which is well known to be generally far more expeditious, even in biquadratics and cubics, than the finite theorems. The following general formula for sursolid equations I have deduced from thence, wherein the method of operation is rendered so simple, that I think none more easy need be wished for.

If $x^5 + ax^4 + bx^3 + cx^2 + dx + m = 0$, then will x be found by this theorem.

$$\frac{4m^{\frac{5}{5}} + 3am^{\frac{4}{5}} + 2bm^{\frac{3}{5}} + cm^{\frac{2}{5}} - m}{d + 5m^{\frac{4}{5}} + 4am^{\frac{3}{5}} + 3bm^{\frac{2}{5}} + 2cm^{\frac{1}{5}}} = x,$$

In which if the value of $m^{\frac{1}{5}}$ being substituted, there is no remainder, the quotient gives one value of x; otherwise, this quotient must be substituted for $m^{\frac{1}{5}}$, and thus repeated till it either terminates or be as near the true root as necessary. The reason of thus assuming $m^{\frac{1}{5}}$ will be obvious, if we consider, that m is always the product of all the roots (having their signs changed) and therefore $m^{\frac{1}{5}}$ will either be a root, or generally near one.

Example 1.

Required the five roots of this equation,

$x^5 - 23x^4 + 159{\cdot}25x^3 - 459x^2 + 564{\cdot}75x - 243 = 0.$

Here $m^{\frac{1}{5}} = 3$, which is affirmative because the signs change alternately; and $4m^{\frac{5}{5}}$ $(=4m) = +972$, $3am^{\frac{4}{5}} = -5589$, $2bm^{\frac{3}{5}} = +8599{\cdot}5$, $cm^{\frac{2}{5}} = -4131$, $-m = +243$; $d = +564{\cdot}75$, $5m^{\frac{4}{5}} = +405$, $4am^{\frac{3}{5}} = -2484$, $3bm^{\frac{2}{5}} = +4299{\cdot}75$, and $2cm^{\frac{1}{5}} = -2754$. Therefore

$$\frac{972 - 5589 + 8599{\cdot}5 - 4131 + 243}{564{\cdot}75 + 405 - 2484 + 4299{\cdot}75 - 2754} = \frac{+94{\cdot}5}{+31{\cdot}5} = +3,$$

a root.

a root. Now dividing the given equation by $x - 3$, we reduce it to this biquadratic,
$x^4 - 20x^3 + 99{\cdot}25x^2 - 161{\cdot}25x + 81 = 0$; the roots of which will be found to be 1, 4, 13·5 and 1·5; hence the five roots of the proposed equation are, 1, 1·5, 3, 4, and 13·5.

Example 2.

Given $x^5 + 15x^4 + 79x^3 + 189x^2 + 208x + 84 = 0$, required the values of x. Here $m^{\frac{1}{5}} = 2{\cdot}424$, &c. which as there are no changes in the signs must be negative, that is $m^{\frac{1}{5}} = -2{\cdot}424$. But in order to shorten the operation, either reject the decimals, or encrease them to unity; suppose the latter, then $m^{\frac{1}{5}} = -3$, and by proceeding as before we find $\frac{-972+3645-4266+1701-84}{405-1620+2133-1134+208} = \frac{+24}{-8} = -3$; which, as it terminates, is one root; and therefore by dividing the given equation by $x + 3$, we get the biquadratic $x^4 + 12x^3 + 43x^2 + 60x + 28 = 0$, whose roots are $-1, -2, -2, -7$, and thence $-1, -2, -2, -3$, and -7, are the roots of the proposed equation.

Note. If in substituting the value of $m^{\frac{1}{5}}$ in the formula, the whole expression should vanish, or become equal 0, there will be two or more roots of the same value in the equation. As in this last Example by rejecting the decimals we have $m^{\frac{1}{5}} = -2$, from whence the formula becomes
$\frac{-128+720-1264+756-84}{208+80-480+948-756} = \frac{0}{0}$; hence there are two values of x, *viz.* -2, equal to each other. The reason of which will be evident to those who are acquainted with the construction of the theorem.

Example 3.

Given $x^5 - 7x^4 + 20x^3 - 155x^2 + 10000 = 0$. Required x. Here $a = -7$, $b = +20$, $c = -155$, $d = 0$, $-m = -10000$, and, by rejecting the decimals, $m^{\frac{1}{5}} = 6$, which may be assumed either affirmative or negative, if we take the latter, then a few operations, by substituting the quotient for $m^{\frac{1}{5}}$, give $x = 4{\cdot}54419572$, which is true to the 7th or 8th place.

Example 4.

Let the radius of a circle be 1, what is the length of the chord of 12 degrees?

The final equation will be $x^5 - 5x^3 + 5x - 1 = 0$. Here $a = 0$, $b = -5$, $c = 0$, $d = +5$, $m = +1$, and $m^{\frac{1}{5}} = 1$, and the formula gives $\frac{4-10+1}{5+5-15} = \frac{-5}{-5} = +1$, which is one root of the equation, but manifestly cannot be that answering the case proposed; agreeable to what was observed in Rem. 2. Therefore assuming $m^{\frac{1}{5}} = {\cdot}2$, the first operation gives $m^{\frac{1}{5}} = {\cdot}209$, and the second $m^{\frac{1}{5}} = {\cdot}209056926$, for the required value of x; or the length of the chord of 12 degrees, when the radius is unity, which is true to the last figure.

13. As it may be of some use to the young Algebraist, I shall now endeavour to illustrate, by a few Examples, the Newtonian method of obtaining the roots of literal equations.

Example 1.

Given, $y^3 + axy - a^2y - x^3 = 0$. Required the value of y, in a series composed of the powers of a and x, with their coefficients.

The

The first thing to be done is to *tabulate* the equation, which is easily effected as follows: Make a right angle DAB (fig. 10.), and from A towards B write down all the powers of y, and from A towards D all the powers of x, as high as in the given equation, beginning from unity. Divide the whole into small squares or parallelograms, as in the scheme, and insert the terms of the equation in their corresponding squares or parallelograms, that is, in those which have the same powers of x and y; then circumscribe the significant parallelograms with the polygon FBCDE, and the equation is tabulated or prepared for extraction.

Now, in order to determine the first term of the series, make any two terms which are placed in two adjacent angles of the polygon equal to 0; that is,

$$y^3 - a^2y = 0,$$
$$- a^2y - x^3 = 0,$$
$$y^3 - x^3 = 0,$$

The two first equations give the first terms of a series for y when x is supposed very little in respect to a; and the last, when a is little in respect to x. We shall take the first, *viz.*

$$y^3 = a^2y,$$

from whence $y = \pm a$; which is the first term of the series.

To obtain the next term, put $y = a + p$, and substitute this value in the given equation, and we shall have,

$$\left.\begin{array}{l} + y^3 = a^3 + 3a^2p + 3ap^2 + p^3, \\ + axy = a^2x + axp, \\ - a^2y = - a^3 - a^2p, \\ - x^3 = - x^3, \end{array}\right\} = 0$$

Select

Select two terms of the greatest value; or, which is the same thing, take two terms where p and x are separately of the lowest dimensions, and make them equal o, from whence p will be had in terms of a and x. Here a, by supposition, is greater than x, and therefore much greater than p; consequently $2a^2p + a^2x$ (that is, $3a^2p - a^2p + a^2x$) must be of greater value than all the other terms of the equation.

From $2a^2p = -a^2x$, we have $p = -\frac{x}{2}$; which is the second term of the series. Put $p = -\frac{x}{2} + q$, and substitute this value in the last equation, omitting all the terms which are higher than that power of x we propose the series to be carried to, putting one term after the next less power of x, two after the next less, and so on. If we propose to carry the root to the fourth power of x, it will stand thus,

$$\left.\begin{array}{rcl} & +p^3 & = -\frac{x^3}{8} + \frac{3qx^2}{4}, \text{ \&c.} \\ & +3ap^2 & = +\frac{3ax^2}{4} - 3aqx + 3aq^2, \\ 3a^2p - a^2p = & +2a^2p & = -a^2x + 2a^2q, \\ & +axp & = -\frac{ax^2}{2} + axq, \\ & +a^2x & = +a^2x, \\ & -x^3 & = -x^3, \end{array}\right\} = 0.$$

Here the terms $\frac{ax^2}{4}\left(= \frac{3ax^2}{4} - \frac{ax^2}{2}\right)$ and $2a^2q$ are vastly greater than the other, q being very little in respect to x, and x very little in respect to a; therefore,

$$\frac{ax^2}{4} = -2a^2q;$$

from

from whence $q = -\frac{x^2}{8a}$, the third term of the series.

Put $q = \frac{x^2}{8a} + r$, and substitute this value in the last equation, and we get,

$$\left.\begin{array}{r}
+3aq^2 = +\frac{3x^4}{6xa}, \&c. \\
+2a^2q = -\frac{ax^2}{4} + 2a^2r, \\
+\frac{3x^2q}{4} = -\frac{3x^4}{32a}, \&c. \\
-3axq + axq = -2axq = +\frac{x^3}{4} - 2axr, \\
\frac{3ax^2}{4} - \frac{ax^2}{2} = +\frac{ax^2}{4} = +\frac{ax^2}{4}, \\
-\frac{x^3}{8} - x^3 = -\frac{9x^3}{8} = -\frac{9x^3}{8},
\end{array}\right\} = 0.$$

The terms to be compared are $-\frac{7x^3}{8}\left(= +\frac{x^3}{4} - \frac{9x^3}{8}\right)$ and $+2a^2r$; that is,

$$2a^2r = \frac{7x^3}{8},$$

from whence $r = +\frac{7x^3}{16a^2}$, the fourth term of the series.

Lastly, let $r = +\frac{7x^3}{16a^2} + s$; this, substituted in the last equation, gives

$-2axr$

$$\left.\begin{array}{l} -2axr = -\frac{7x^4}{8a}, \text{ \&c.} \\ +2a^2r = +\frac{7x^3}{8} + 2a^2s, \\ -\frac{7x^3}{8} = -\frac{7x^3}{8}, \\ -\frac{3x^4}{64a} = -\frac{3x^4}{64a}, \end{array}\right\} = 0.$$

Here all the terms vanish but $2a^2s$, and $-\frac{59x^4}{64a}$ $\left(= -\frac{7x^4}{8a} - \frac{3x^4}{64a}\right)$; which, being therefore compared as before, *viz.*

$$2a^2s = \frac{59x^4}{64a},$$

we have $s = +\frac{59x^4}{128a^3}$, the fifth term of the series; so $y = a - \frac{x}{2} - \frac{x^2}{8a} + \frac{7x^3}{16a^2} + \frac{59x^4}{128a^3}$, &c.

It will easily be perceived, by a proper attention to the method of operation, that every new term in the series is had by dividing the lowest of the terms affected with the indefinitely small quantity x, or its powers, without the assumed ones, p, q, r, &c. by the quantity with which the assumed one of one dimension only is multiplied; thus, the second term is had by dividing a^2x by $2a^2$; the third by dividing $\frac{ax^2}{4}$ by $2a^2$, &c. From which it is evident, that when the work is continued so far, as that the assumed quantity (p, q, r, &c.) is only of one dimension, the remaining terms of the root may be had by one division; so, when $-\frac{x^2}{8a} + r$ is substituted for y, it appears

appears that r is only of one dimension, therefore the remaining two terms will be had thus,

$$2a^2 - 2ax)\frac{7x^3}{8} + \frac{3x^4}{64}\left(+ \frac{7x^3}{16a^2} + \frac{59x^4}{128a^3}\right.$$

$$\frac{7x^3}{8} - \frac{7x^4}{8a}$$

$$+ \frac{59x^4}{64a}$$

$$+ \frac{59x^4}{64a}, \&c.$$

Example 2.

Given $\frac{y^5}{5} - \frac{y^4}{4} + \frac{y^3}{3} - \frac{y^2}{2} + y + a - x = 0$. Required y in terms of a and x, which are here supposed to be nearly equal.

In order then to make the series properly converge, we must substitute for their difference, that is, put $x = a + z$, and the given equation will become

$$\frac{y^5}{5} - \frac{y^4}{4} + \frac{y^3}{3} - \frac{y^2}{2} + y - z = 0.$$

This being tabulated (as in fig. 11.) we have $y = z$, for the first term of the series. Put $y = z + p$, and write this value in the above equation, and there arises,

$$\left.\begin{array}{lcl} +\frac{1}{5}y^5 & = & * \\ -\frac{1}{4}y^4 & = & -\frac{1}{4}z^4, \&c. \\ +\frac{1}{3}y^3 & = & +\frac{1}{3}z^3 + z^2p \\ -\frac{1}{2}y^2 & = & -\frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\ +\ y & = & +z + p \\ -\ z & = & -z \end{array}\right\} = 0.$$

The terms to be equated are p, and $-\frac{1}{2}z^2$, from which the second term of the series is $+\frac{1}{2}z^2$. Put $p=\frac{1}{2}z^2+q$, and substitute this in the last equation for p, and it becomes,

$$\left.\begin{array}{lcl} -\frac{1}{2}p^2 & = & -\frac{1}{8}z^4,\ \&c. \\ +z^2p & = & +\frac{1}{2}z^4,\ \&c. \\ -zp & = & -\frac{1}{2}z^3-zq. \\ +p & = & +\frac{1}{2}z^2+q \\ -\frac{1}{4}z^4 & = & -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 & = & +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 & = & -\frac{1}{2}z^2 \end{array}\right\} = 0.$$

Here the terms to be compared are q and $-\frac{z^3}{6},\left(=-\frac{z^3}{2}+\frac{z^3}{3}\right)$ but as q is only of one dimension, the remaining terms may be found by division, thus;

$$\begin{array}{l} 1-z\Big)+\frac{z^3}{6}-\frac{z^4}{8}\Big(+\frac{z^3}{6}+\frac{z^4}{24}. \\ \qquad\quad +\frac{z^3}{6}-\frac{z^4}{6} \\ \hline \qquad\qquad\quad +\frac{z^4}{24} \\ \qquad\qquad\quad +\frac{z^4}{24},\ \&c. \\ \hline \end{array}$$

Therefore the root is $y=z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4$, &c.; or, by restoring the value of x,

$$y=x-a+\frac{\overline{x-a}|^2}{2}+\frac{\overline{x-a}|^3}{3\cdot 2}+\frac{\overline{x-a}|^4}{3\cdot 4\cdot 2},\ \&c.$$

where the law is manifest, and may be continued at pleasure, *viz.*

$$+\frac{\overline{x-a}|^5}{3\cdot 4\cdot 5\cdot 2}+\frac{\overline{x-a}|^6}{3\cdot 4\cdot 5\cdot 6\cdot 2},\ \&c.$$

It

It appears from the table that other series may be found from the equations,

$$\tfrac{1}{5}y^5 = z, \text{ and } \tfrac{1}{5}y^4 - \tfrac{1}{4}y^3 + \tfrac{1}{3}y^2 - \tfrac{1}{2}y = -1.$$

Remark. Sir Isaac, or some of his transcribers, seem to have committed an oversight in supposing we might put either $a + z$, or $a - z$ for x, for it is indispensably necessary that x should be greater than a, otherwise the series will not converge. The substituting of $a - z$ for x gives the last term of the given equation affirmative, and from thence the value of y comes out $z + 2z + 4z + 8z + 16z$, &c. which will evidently be a *diverging* series, however small we suppose z to be. Therefore, in order to have a *converging* series for y, when a is greater than x, we must change the species, that is, write a for x, and x for a. So if $a - x = z$, a will be equal $z + x$; this substituted in the given equation, gives the last term negative as before, therefore the root will be

$$y = a - x + \frac{\overline{a-x}|^2}{2} + \frac{\overline{a-x}|^3}{3.2} + \frac{\overline{a-x}|^4}{3.4.2}, \text{ \&c.}$$

Example 3.

Given $y^3 + y^2 + y - x^3 = 0$. Required y in terms of x. Here it is evident that x must be very great, and therefore the common method cannot exhibit a true value of y. But to make the series converge, put $\frac{1}{x} = z$, and thence will $x = \frac{1}{z}$. Substitute this for x in the given equation, and there arises $y^3 + y^2 + y - \frac{1}{z^3} = 0$. Tabulate this equation (fig. 12.) and the terms to be compared will be y^3 and $\frac{1}{z^3}$,

from whence $y = \frac{1}{z}$, the first term of the series, and proceeding as before, the root is found

$$= \frac{1}{z} - \frac{1}{3} - \frac{2}{9}z + \frac{7}{81}z^2 + \frac{5}{81}z^3, \text{ \&c.}$$

Restoring x it becomes $y = x - \frac{1}{3} - \frac{2}{9x} + \frac{7}{81x^2}$, &c.

The other equations are $y = \frac{1}{z^3}$, and $y^2 + y = -1$. The former gives $y = \frac{1}{z^3} - \frac{1}{z^6} + \frac{1}{z^9}$, &c. which, as it ascends in the powers of z in the denominators, must also ascend in the powers of x in the numerators; and therefore will evidently diverge; and the latter gives $y = \sqrt{-\frac{3}{4}} - \frac{1}{2}$, impossible.

Example 4.

Given $y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \frac{35}{1152}y^9 + \frac{63}{2816}y^{11}$, &c. $- x = 0$. Required x in terms of y. Here y, and consequently x, must be supposed less than unity, in order that the series may duly converge. Having tabulated the equation (fig. 13.) we have $y = x$, the first term of the series. Put $y = x + p$, and substitute this value of y in as many terms of the given equation as the series is proposed to be carried to; suppose to four terms, and the operation will stand as follows,

$$\begin{aligned}
&+ \frac{63}{2816}y^{11} &&= * \\
&+ \frac{35}{1152}y^9 &&= * \\
&+ \frac{5}{112}y^7 &&= + \frac{5}{112}x^7, \text{ \&c.} \\
&+ \frac{3}{40}y^5 &&= + \frac{3}{40}x^5 + \frac{3}{8}x^4p, \text{ \&c.} \\
&+ \frac{1}{6}y^3 &&= + \frac{1}{6}x^3 + \frac{1}{2}x^2p + \frac{1}{2}xp^2, \text{ \&c.} \\
&+ y &&= + x + p \\
& && - x
\end{aligned}$$

Hence

Fig. 10

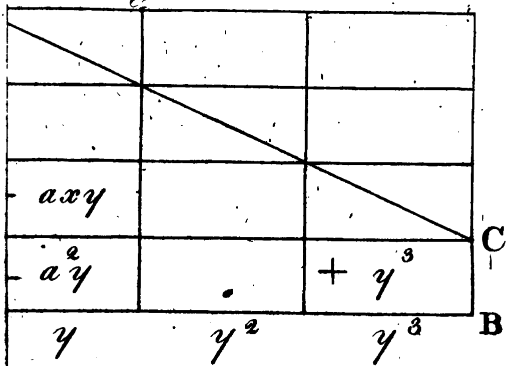


Fig. 11

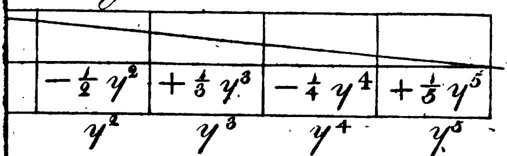


Fig. 12

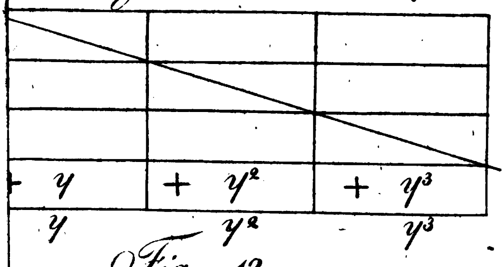
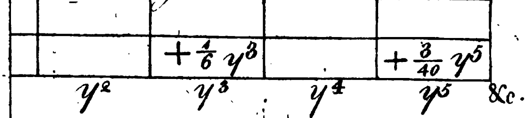


Fig. 13



Hence $p = -\frac{1}{6}x^3$. Put $p = -\frac{1}{6}x^3 + q$, and substitute this in the last equation, and it becomes,

$$\left.\begin{array}{l}
+\frac{1}{2}xp^2 = +\frac{1}{72}x^7, \text{ \&c.} \\
+\frac{3}{8}x^4p = -\frac{1}{16}x^7, \text{ \&c.} \\
+\frac{1}{2}x^2p = -\frac{1}{12}x^5 + \frac{1}{2}x^2q \\
+\quad p = -\frac{1}{6}x^3 + q \\
\qquad\quad +\frac{5}{112}x^7 \\
\qquad\quad +\frac{3}{40}x^5 \\
\qquad\quad +\frac{1}{6}x^3
\end{array}\right\} = \left\{\begin{array}{l}
-\frac{1}{252}x^7, \text{ \&c.} \\
\\
-\frac{1}{120}x^5 + \frac{1}{2}x^2q \\
\\
+q
\end{array}\right\} = 0.$$

Here q is of one dimension only, therefore,

$$1 + \frac{x^2}{2}\Big) + \frac{1}{120}x^5 + \frac{1}{252}x^7\Big(+ \frac{1}{120}x^5 - \frac{1}{5040}x^7, \text{ \&c.}$$

$$\underline{+\frac{1}{120}x^5 + \frac{1}{240}x^7}$$

$$-\frac{1}{5040}x^7$$

$$\underline{-\frac{1}{5040}x^7, \text{ \&c.}}$$

Hence the root is $y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$, &c. or, $y = x - \frac{x^3}{6} + \frac{x^5}{4.5.6} - \frac{x^7}{4.5.6.7.6} + \frac{x^9}{4.5.6.7.8.9.6} - \frac{x^{11}}{4.5.6.7.8.9.10.11.6}$, &c. the law of continuation being evident.

 This

This last example is usually called the *reversion of a series*. There are several other methods of performing it; but this, I think, is as easy to be understood as any of them. The young student may here see with what facility these intricate affairs are managed when freed from all that unnecessary prolixity we find them embarrassed with in most authors. What before seemed almost insuperable to him may be now only a pleasant and agreeable exercise, as he cannot fail of understanding the method of operation, if he but attentively examine these examples. Indeed, there may be a great many literal equations proposed which may not coincide with any of the examples we have given. But when the equation is tabulated, and two or three of the first terms of the series obtained, it will immediately appear whether it properly converges or no. If all the equations diverge, or have impossible roots, a little artifice must be used, such as augmenting or diminishing the roots by some known quantity; or by taking the reciprocal of an indefinitely large quantity, and so on. So that in almost all cases the series may be made to converge, and the root obtained by the foregoing method.

Some useful Remarks on the nature and method of Fluxions.

As I have always found the following remarks of service to the learner, I shall make no apology for inserting them in this treatise, which, I have before observed, is purely designed as an help to the young reader, in removing some of those little obstacles which he must unavoidably meet with at his first entrance on these studies.

The doctrine of *prime and ultimate ratios*, by which the fluxions of quantities are generally investigated, or demonstrated, contains in it something so very obscure and unintelligible

telligible to the learner, that it is rather more apt to confuse than give a proper arrangement to his ideas on the subject *.

The

* The first Lemma of Sir Isaac Newton's Principia appears to many to be very exceptionable; his words are,—*Quantitates, ut et quantitatum rationes, quæ ad æqualitatem tempore quovis finito constanter tendunt, et ante finem temporis illius propius ad invicem accedunt quam pro data quavis differentia, fiunt ultimo æquales.* And then adds, *Si negas; fiant ultimo inæquales, et sit earum ultima differentia* D. *Ergo nequeunt propius ad æqualitatem accedere quam pro data differentia* D: *contra hypothesin.* Now, though this method of demonstration is far from being satisfactory, being a kind of reasoning *mutatis mutandis*; yet, it is plain that the position may be readily admitted if the *decrements* only, or *parts destroyed*, are to be proportional. For instance, suppose two lines, the one 20 inches, the other 12, to be diminished by some cotemporary decrements, of which the ratio is respectively as 3 to 1. Here then it is evident, that the quantities 20 and 12, would soon become equal by such a diminution. Thus, in the first portion of time, let the former line be reduced to 14; then the latter will become 10. In the second portion of time, the former becomes 8, and also the latter 8. They have therefore converged to equality by a cotemporary diminution with proportional decrements. But when the quantities themselves are supposed to be proportionably diminished, and thereby to obtain the ratio of equality, I think it will appear that there is nothing more absurd. For suppose the quantities 20 and 12 to be any-how proportionably diminished, suppose by a continual bisection, and it is evident, that they will converge to equality in respect of their difference, and yet retain their original proportion. Thus, the first difference of the proposed quantities 20 and 12, is 8 inches; the first bisection reduces that difference to 4 inches, and the quantities themselves to 10 and 6; but 10 is to 6, as 20 is to 12; therefore the proportion is not altered. The second bisection makes the difference only 2 inches; but still the quantities are in the same ratio, for 5 is to 3, as 20 to 12. The third bisection reduces the difference to 1 inch, and the quantities themselves to $2\frac{1}{2}$ and $1\frac{1}{2}$, which are still in the same proportion. Hence then it plainly appears, that two quantities may converge to equality in respect of their difference, and that *that* difference may become less than any assignable quantity, yet the quantities themselves can never become equal; for in whatever ratio they were originally, in the same ratio will they remain, if

 diminished

The most natural and easy way of acquiring a right notion of fluxions, is by the introducing of *time* into the account. For by this means we do not consider them as mere velocities, which

diminished *fine fine*, according to the mathematical doctrine of the infinite divisibility of matter.

How absurd then must it appear, to attempt to find the last ratio of the cotemporary increments of two flowing quantities by a continual diminution of those increments, since it is obvious they will always remain in the same ratio, however small they are taken. And yet after this manner have Colson, Ditton, Hayes, and several other writers on fluxions, pretended to find the last ratio of the vanishing increments. Thus if the increment of x be denoted by $\acute{x}$, the cotemporary increment of x^n will be expressed by $nx^{n-1}\acute{x} + \frac{n^2-n}{2}x^{n-2}\acute{x}^2 + \frac{n^3-3n^2+2n}{2.3}x^{n-3}\acute{x}^3$, &c. Here then, say they, by continually diminishing these increments, we approach nearer to the ratio with which they first arise or begin to be generated; and therefore, when $\acute{x}$ is become infinitely small, the higher powers of it must vanish first, and leave the infinitely small increments in the ratio of $\acute{x}$ to $nx^{n-1}\acute{x}$; which is therefore their *prime* or *ultimate ratio*. But does this appear in the least satisfactory? Is it not rather a mere quibble? For if the increment of x have a *real* value, though ever so small, it is obvious, that the increment of x^n cannot be accurately $nx^{n-1}\acute{x}$. And if $\acute{x}^2$, $\acute{x}^3$, &c. be absolutely *nothing*, certainly the root itself must be *nothing* also; and consequently the whole expression must vanish together. Besides, this is plainly contradictory to the lemma; for, if that be admitted, the last ratio must be that of equality and not of 1 to nx^{n-1}, since the quantities are proportionably diminished in the same time.

Some other writers, who, I imagine, saw the fallacy of this method of reasoning, have endeavoured to obviate these difficulties, by representing the affair in a different light, as follows: If we consider that the increment of x^n is much different from the fluxion of it, the former being described by an accelerated motion, and the latter by an uniform one, it will not be so hard to conceive, that by continually diminishing the increment of the simple quantity x, the increment of the compound quantity x^n will come nearer

which naturally involve the mind in metaphyfical difficulties; but as the magnitudes they uniformly generate in a given finite time, fuppofing the fluent or fpace to be defcribed by an uniform motion. And if the motion by which any magnitude is generated be not uniform, but accelerated or retarded, the idea of a fluxion will ftill be the fame: For though we cannot exprefs the fluxion by any fpace *actually* generated in a given time, as in uniform motion; yet we can readily affign the magnitude, or (as it is commonly called) the cotemporary increment, that *would be* uniformly generated, if the acceleration or retardation were to ftop at any point in which the fluxion is required to be inveftigated.

Now, as our ideas of magnitude arife from a comparifon of the propofed object with fome other of determinate dimenfions: fo, in the method of fluxions, we fix on a given magnitude, which is fuppofed to have been uniformly generated in a given time by the motion of a point, line, or plane, as a ftandard,

nearer to an uniform velocity; and therefore, juft in the inftant of vanifhing, or when it becomes *nothing*, the velocity muft be uniform, and truly exprefs the fluxion at that point. Hence then it appears, that it is by confounding the increment and fluxion, that thefe feeming abfurdities arife. For if we imagine the fluxion of the *uniformly* generated quantity to be an infinitely fmall, yet certain magnitude, it will readily appear, by keeping our ideas of the fluxion and increment diftinct, how the ratio of the fluxions may be had, when the increments themfelves vanifh, or become *nothing*. Thus, let $\dot{x}$ denote the fluxion of the fimple quantity x, or that infinitely fmall quantity which would be uniformly defcribed by the generating point in one inftant of time; then, fince the ratio of the increments, $x' : nx^{n-1}x' + \frac{n^2-n}{2}x^{n-2}x'^2$, &c. is the fame as $1 : nx^{n-1} + \frac{n^2-n}{2}x^{n-1}x'$, &c. or, by multiplying by $\dot{x}$, as $\dot{x} : nx^{n-1}\dot{x} + \frac{n^2-n}{2}x^{n-2}\dot{x}x'$, &c. it will be evident, that if we fuppofe the increment of x to vanifh, the ratio will then become accurately, $\dot{x} : nx^{n-1}\dot{x}$. But this is too metaphyfical for moft readers.

wherewith

wherewith to compare any other magnitude, which is ſuppoſed to have been generated in the ſame time, by an accelerated or retarded motion. Thus (for the ſake of illuſtration) ſuppoſe a ball to roll on an horizontal plane, in a ſtraight direction, at the uniform rate of 20 feet in a minute; and alſo another ball to move uniformly in the ſame direction, at the rate of 40 feet in the ſame time; here then it will be plain, that the magnitudes generated in any given time muſt be in the ratio of 2 to 1; and therefore the fluxion of the latter will be double that of the former. And from hence it appears, that if the fluxion of x be $\dot{x}$, that of $2x$ will be $2\dot{x}$, $3x$ will be $3\dot{x}$, &c. and generally, that of nx will be $n\dot{x}$. But if, while one ball moves along with an uniform velocity, the other is ſuppoſed to move with an accelerated motion, and that the law of acceleration is ſuch, that the ſpace deſcribed by the latter, from the commencement of motion, is always ſome power of that deſcribed by the former, ſuppoſe the ſquare of it; then, if the magnitude by which the ſpace that is uniformly deſcribed is increaſed in a given time be denoted by $\dot{x}$, that magnitude which the accelerated motion would uniformly generate in the ſame time, and commencing from the ſame inſtant, will be expreſſed by $2x\dot{x}$. Thus, in the caſe propoſed, if the firſt ball has uniformly deſcribed a ſpace of 10 poles, the other muſt have run 100 poles; but the former ball moves uniformly at the rate of 20 feet in a minute, therefore the magnitude or ſpace, which the accelerated ball would uniformly deſcribe from the ſame inſtant in one minute, will be 400 feet. The fluxions will be therefore at that point in the ratio of 400 to 20; or of 20 to 1. This is eaſily demonſtrated as follows: Let $x - t$, and x, repreſent the ſpaces deſcribed by the uniform motion at any two given times, then (by Hypotheſis) will the cotemporary ſpaces, paſſed over by the accelerated motion, be $(\overline{x-t}|^2 =) \; x^2 - 2xt + t^2$, and x^2; and

therefore $x^2 - \overline{x^2 - 2xt + t^2} = 2\dot{x}t - t^2$ will be the difference of those spaces. From whence it follows, that, while the first ball runs uniformly over the space t, the other runs over the space $2xt - t^2$. Now, as this space is not generated by an uniform but accelerated motion, it cannot represent the fluxion of either of the spaces expressed by $\overline{x - t}|^2$ or x^2; but it may evidently represent the magnitude which might be uniformly described with the mean velocity at some point between $\overline{x - t}|^2$ and x^2. But this magnitude is to that generated in the same time by the uniform motion as $2xt - t^2$ is to t; or, by multiplying by $\frac{\dot{x}}{t}$ as $2x\dot{x} - t\dot{x}$ to $\dot{x}$; therefore, when $x - t$ by the uniform motion of the ball becomes x, t vanishes, or is equal to *nothing*, and consequently the point of mean velocity then coincides with x; and hence the above ratio becomes barely, as $2x\dot{x} : \dot{x}$. Therefore, according to the common phrase, the fluxion of x^2 is $2x\dot{x}$.

This may be generally represented thus: Let the law of acceleration (or retardation, prefixing the negative sign) be universally expressed by x^n, and by the same method of reasoning, by help of the binomial theorem, we shall find the cotemporary fluxions to be in the ratio of $nx^{n-1}\dot{x}$ to $\dot{x}$. For if $x - t$ and x represent the spaces uniformly described as before, the cotemporary spaces described by the accelerated motion will be expressed by $\overline{x - t}|^n$ and x^n; but the difference of these spaces $x^n - \overline{x - t}|^n$ is equal $nx^{n-1}t - \frac{n^2 - n}{2} . x^{n-2}t^2$, &c. therefore the ratio of the magnitudes, generated in the same time by the uniform motion and the mean velocity, will be expressed by $nx^{n-1}t - \frac{n^2 - n}{2} . x^{n-2}t^2$, &c. to t; or, by multiplying by $\frac{\dot{x}}{t}$, as $nx^{n-1}\dot{x} - \frac{n^2 - n}{2} . x^{n-2}t\dot{x}$, &c. to $\dot{x}$;

and

and when $x-t$ is equal x, t will evidently be $=0$, hence all the terms wherein t is found must vanish, and the point of mean velocity coincide with x; consequently the ratio of the fluxions will be as $nx^{n-1}\dot{x}$ to $\dot{x}$.

Hence then it appears, that we have the most rational notion of fluxions from the consideration of time in the generation of the increment or decrement, and that the fluxion of any variable quantity may be truly defined, *The magnitude by which any flowing quantity would be increased in a given time with the generating velocity at a given instant, supposing it from thence to proceed uniformly or invariably.* And with regard to the higher orders of fluxions, how much more obscure are our notions without the idea of time in the operation of the fluent generating the increment; since by having recourse to the first ratio of the nascent increment, or the last ratio of the evanescent increment, even to obtain only the first fluxion of a variable quantity, we unavoidably fall into this absurdity, *That a velocity which continues for no time at all actually describes a space.* How then can we form any conception not only of such a space or increment, but also of an infinite variety of magnitudes of it, generated in one and the same point and instant of time, in which it is well known all the orders of fluxions are considered, when nothing, I think, can be more evident than that the magnitude or increment imagined to be generated must in such a case be *purum putum nihil*, or strictly and absolutely nothing. If a doubt of the existence of an increment under such circumstances be deemed incredulity and a species of infidelity*, I am afraid I shall be stigmatized with those appellations; for I confess it is past my comprehension how a mere point can contain in itself an infinite variety of magnitudes, and which are all at the same time

* See Colson's Newton's Flux. p. 18, Preface.

equal

equal to one another. These unnecessary quibbles, and metaphysical niceties, by which some have attempted to explain the principles of fluxions, have not only rendered them quite obscure to the learner, but also exposed them to the ridicule and severe criticisms of several writers of great abilities in the mathematics. But these criticisms, it is probable, were not intended to invalidate the method of fluxions (which it is evident may be strictly mathematically demonstrated) but to shew the futility of the method they had taken to elucidate the principles; in which light it is well known the incomparable inventor never intended they should be viewed.

From what has been said we may draw these practical observations.

1. That the common rule for finding the fluxion of a flowing quantity, *viz. Multiply the fluxion of the root by the exponent of the power and the affixed coefficient, and the product by that power of the same root of which the exponent is less by unity than the given exponent*, is general, and without exceptions, being applicable to any expression whatever consisting of one variable quantity with a constant exponent. 2. If the expression be a compound one, that is a binomial, trinomial, or any multinomial, the fluxion of each term must be found separately, and connected with their respective resulting signs; the sum arising by such addition is the fluxion of the compound expression. 3. If the expression consists of the product of two or more variable quantities, each quantity must flow separately, while the others are supposed to be constant, or as coefficients to that variable quantity; the sum of these fluxions will be that of the given expression. This follows from the general expression $nx^{n-1}\dot{x}$. Thus, let the fluxion of yz be proposed to be investigated. Put $y+z=v$, then will $y^2+2yz+z^2=v^2$; hence

hence $yz = \frac{v^2 - y^2 - z^2}{2} = \frac{1}{2}v^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2$. And, from what has been before ſhewn, the fluxion of this will be $v\dot{v} - y\dot{y} - z\dot{z}$; but $v = y + z$, and $\dot{v} = \dot{y} + \dot{z}$, ∵ by ſubſtitution, the fluxion of yz is $y\dot{z} + z\dot{y}$. And in the ſame manner will the fluxion of xyz be found to be $xy\dot{z} + xz\dot{y} + yz\dot{x}$; and that of $uxyz = uxy\dot{z} + uxz\dot{y} + uyz\dot{x} + xyz\dot{u}$; &c.

We ſhall now give a few examples in order to ſhew the propriety of theſe remarks.

Let the fluxion of x be $\dot{x}$; that is, let the magnitude or ſpace which is paſſed over in a given time by the uniform motion of a point in generating the ſpace x be denoted by $\dot{x}$; then will the fluxions, or relative magnitudes, which would be uniformly generated in the ſame time, of the cotemporary fluents, or ſpaces already paſſed over when the fluxions are compared, be obtained by the general rule as follows,

Variable

Variable Quantities.	*Fluxions.*
1. $2x$.	$1 \times 2x^{1-1}\dot{x} = 2\dot{x}$ *.
2. nx,	$1 \times nx^{1-1}\dot{x} = n\dot{x}$.
3. x^3,	$3x^{3-1}\dot{x} = 3x^2\dot{x}$.
4. $5x^7$,	$7 \times 5x^{7-1}\dot{x} = 35x^6\dot{x}$.
5. mx^n,	$mnx^{n-1}\dot{x}$.
6. $\frac{m}{n}x^{\frac{t}{r}}$,	$\frac{t}{r} \times \frac{m}{n}x^{\frac{t}{r}-1}\dot{x}, = \frac{tm}{rn}x^{\frac{t-r}{r}}\dot{x}$.
7. $\frac{m}{n}x^{\frac{t}{r}}y^{\frac{p}{q}}$,	$y^{\frac{p}{q}} \times \frac{m}{n} \times \frac{t}{r}x^{\frac{t}{r}-1}\dot{x} + x^{\frac{t}{r}} \times \frac{m}{n} \times \frac{p}{q}y^{\frac{p}{q}-1}\dot{y}, = \frac{mt}{nr}y^{\frac{p}{q}}x^{\frac{t-r}{r}}\dot{x} + \frac{mp}{nq}x^{\frac{t}{r}}y^{\frac{p-q}{q}}\dot{y}$.

* That x^{1-1} is equal to 1, will be evident, if we consider that x^{1-1} is $= x^1 \times x^{-1}$; which is equal $\frac{x^1}{x^1} = 1$. And therefore, generally, $x^{n-n} = \frac{x^n}{x^n} = 1$.

Variable

Variable Quantities.	*Fluxions.*
8. $\frac{x}{y}, = xy^{-1},$	$x \times -y^{-2}\dot{y} + y^{-1}\dot{x}, = \frac{y\dot{x} - x\dot{y}}{yy}.$
9. $\frac{1}{\sqrt[n]{x}}, = x^{-\frac{1}{n}},$	$-\frac{1}{n} x^{-\frac{1+n}{n}} \dot{x}, = -\frac{\dot{x}}{n x^{\frac{1+n}{n}}}.$
10. $\sqrt{a^2 - x^2}, = \overline{a^2 - x^2}\vert^{\frac{1}{2}},$	$\frac{1}{2} \cdot \overline{a^2 - x^2}\vert^{-\frac{1}{2}} \times -2x\dot{x}, = -\frac{x\dot{x}}{\sqrt{a^2 - x^2}}.$
11. $a\sqrt{x^2 + y^2}, = a. \overline{x^2 + y^2}\vert^{\frac{1}{2}},$	$\frac{1}{2} a \times \overline{x^2 + y^2}\vert^{-\frac{1}{2}} \times \overline{2x\dot{x} + 2y\dot{y}}, = \frac{ax\dot{x} + ay\dot{y}}{\sqrt{x^2 + y^2}}.$
12. $\sqrt[\frac{m}{n}]{y}, = y^{\frac{n}{m}}$ *,	$\frac{n}{m} y^{\frac{n-m}{m}} \dot{y}, = \frac{n\dot{y}}{my^{\frac{m-n}{m}}}.$

* It may, perhaps, appear ſtrange to the young reader, that $\sqrt[\frac{m}{n}]{y}$ ſhould be $= y^{\frac{n}{m}}$. But, if he is acquainted with the nature of logarithms, he need not be informed, that to extract the $\frac{m}{n}$th root is to divide the log. of y by $\frac{m}{n}$; but it is eaſily proved that to divide by $\frac{m}{n}$ is the ſame as to multiply by its reciprocal $\frac{n}{m}$; therefore $\sqrt[\frac{m}{n}]{y} = y^{\frac{n}{m}}$. Thus, if $m = 1$, and $n = 2$; $\sqrt[\frac{1}{2}]{y}$ will be equal y^2. For dividing the log. of y by $\frac{1}{2}$, is evidently the ſame as multiplying it by 2.

Variable

Variable Quantities.	*Fluxions.*
13. $\sqrt[\frac{m}{n}]{yx} = y^{\frac{n}{m}} x^{\frac{n}{m}}$,	$\dfrac{n x^{\frac{n}{m}} \dot{y}}{m y^{\frac{m-n}{m}}} + \dfrac{n y^{\frac{n}{m}} \dot{x}}{m x^{\frac{m-n}{m}}}$.
14. $\sqrt{y^m z^n} = y^{\frac{m}{2}} z^{\frac{n}{2}}$,	$\dfrac{n\sqrt{y^m}\,\dot{z}}{2z^{\frac{2-n}{2}}} + \dfrac{m\sqrt{z^n}\,\dot{y}}{2y^{\frac{2-m}{2}}}$.
15. $\frac{m}{n}\sqrt[\frac{r}{t}]{x^{\frac{p}{q}} + y^{\frac{v}{w}}}$,	$\dfrac{\frac{p}{q} x^{\frac{p-q}{q}} \dot{x} + \frac{v}{w} y^{\frac{v-w}{w}} \dot{y}}{\frac{tn}{rm}\sqrt[\frac{r}{r-t}]{x^{\frac{p}{q}} + y^{\frac{v}{w}}}}$.
16. $z\sqrt[n]{x^m y^n + z^t}$,	$\dfrac{x^m y^n z^{n-1}\dot{z} + x^m z^n y^{n-1}\dot{y} + \frac{m}{n} y^n z^n x^{m-1}\dot{x} + \frac{t+n}{n} z^{t+n-1}\dot{z}}{\sqrt[\frac{n}{n-1}]{\overline{x^m y . z}^n + z^{t+n}}}$.

K

Variable

Variable Quantities. | *Fluxions.*

And by the same rule are the fluxions of logarithms found, only observing to put down the number itself under the fluxion, as a denominator.

Variable Quantities.	Fluxions.
17. $L.\sqrt[q]{\overline{x^m + y^n}}$.	$\dfrac{\frac{1}{q}\,\overline{x^m + y^n}\rceil^{\frac{1-q}{q}} \times \overline{mx^{m-1}\dot{x} + ny^{n-1}\dot{y}}}{\sqrt[q]{x^m + y^n}} = \dfrac{mx^{m-1}\dot{x} + ny^{n-1}\dot{y}}{q \times \overline{x^m + y^n}}$.
18. $L^n.\overline{x+y}\rceil^m$.	$nL^{n-1}.\overline{x+y}\rceil^m \times \dfrac{m.\overline{x+y}\rceil^{m-1} \times \overline{\dot{x}+\dot{y}}}{\overline{x+y}\rceil^m} = nL^{n-1}.\overline{x+y}\rceil^m \times \dfrac{m\dot{x}+m\dot{y}}{x+y}$.

And from hence may the fluxion of any quantity, of which the index is variable, be easily found.

Thus, the fluxion of A^z is $A^z L.A\dot{z}$. For if we put $A^z = v$, then will $\dot{v}$ = the fluxion of A^z. Take the logarithm on both sides, and we have $L.A^z$ = (from the nature of logarithms) $zL.A = L.v$, ∴ $\dot{z}L.A = \dfrac{\dot{v}}{v}$, and $vL.A\dot{z} = \dot{v}$; hence $A^z L.A\dot{z}$ is the fluxion of A^z.

If

If the quantity and index are both variable, the fluxion may be had after the ſame manner: Suppoſe the fluxion of $\overline{x-y}|^{z^v}$ was required, where all the quantities are variable. The operation will ſtand thus, $\overline{x-y}|^{z^v} = w$; $z^v \mathrm{L}.\overline{x-y} = \mathrm{L}.w$; $vz^{v-1}\dot{z}\,\mathrm{L}.\overline{x-y} + z^v \mathrm{L}.z\dot{v}\,\mathrm{L}.\overline{x-y} + z^v \times \frac{\dot{x}-\dot{y}}{x-y} = \frac{\dot{w}}{w}$;

$\therefore \overline{x-y}|^{z^v} \times \overline{vz^{v-1}\dot{z}\,\mathrm{L}.\overline{x-y} + z^v \mathrm{L}.z\dot{v}\,\mathrm{L}.\overline{x-y} + z^v \times \frac{\dot{x}-\dot{y}}{x-y}}$, is the fluxion of $\overline{x-y}|^{z^v}$.

In the *inverse method of fluxions*, nothing can be more eaſy than to proceed in a retrograde order from the fluxion to the fluent (in expreſſions that will admit of it) by the converſe of the general rule, namely, *Strike out the fluxionary letter, add* one *to the index, and divide by the index ſo increaſed.*

Example.

Required the fluent of $nx^{n-1}\dot{x}$. The proceſs will ſtand thus; $nx^{n-1}\dot{x}$, nx^{n-1}, nx^{n-1+1}, nx^{n}, $\frac{nx^{n}}{n}$, x^{n}, the fluent.

In expreſſions affected with a radical, if the fluxion without the radical, or vinculum, be the fluxion of the quantity under it, the fluent may be obtained by the general rule, obſerving to ſtrike out, not the fluxionary letter *only*, but *all* that part which appears to be the fluxion of the quantity under the radical.

Example.

Required the fluent of $\sqrt[\frac{1}{n}]{a^{m} + x^{m}} \times mx^{m-1}\dot{x}$. The proceſs will be as follows;

$\overline{a^{m} + x^{m}}\rceil^{n} \times mx^{m-1}\dot{x}$, $\overline{a^{m} + x^{m}}\rceil^{n}$, $\overline{a^{m} + x^{m}}\rceil^{n+1}$; $\frac{\overline{a^{m} + x^{m}}\rceil^{n+1}}{n+1}$, the fluent.

If the fluxion without the radical be not that of the quantity under it, but in a *given* ratio to it, the fluent may ſtill be had by the general rule; obſerving to increaſe or decreaſe it in that given ratio.

Example.

Required the fluent of $\overline{a + bx^{m}}\rceil^{n} \times cx^{m-1}\dot{x}$.

The

The fluxion of bx^m is $bmx^{m-1}\dot{x}$; therefore the ratio of the fluxion of the quantity under the radical is to that without it, as $bm : c$. Hence the process will be thus;

$$\overline{a+bx^m}|^n \times cx^{m-1}\dot{x},\ \overline{a+bx^m}|^n,\ \overline{a+bx^m}|^{n+1},\ \frac{\overline{a+bx^m}|^{n+1}}{n+1};$$

then, as $bm : c :: \dfrac{\overline{a+bx^m}|^{n+1}}{n+1} : \dfrac{\overline{a+bx^m}|^{n+1} \times c}{bm \times \overline{n+1}}$, the fluent.

If the fluxion without the radical be neither the fluxion of the quantity contained under it, nor in any given ratio to it, yet, in many forms of expressions, the fluents may still be had in finite terms, by help of the binomial theorem, which will always terminate, when a number equal to the exponent of the flowing quantity without the radical comes to be subtracted from it.

Example.

Required the fluent of $x^n\dot{x}\sqrt[m]{a+x}$.

Here the ratio of the fluxion of the quantity under the radical is to that without it, as $1 : x^n$, and consequently not given; therefore the easiest way is to proceed by substitution, as follows; put $a+x=v$, then will $x=v-a$, $x^n=\overline{v-a}|^n$, and $\dot{x}=\dot{v}$, $\therefore x^n\dot{x}\sqrt[m]{a+x}=v^{\frac{1}{m}}\dot{v}\times\overline{v-a}|^n$. But by the binomial theorem, $\overline{v-a}|^n=v^n-nv^{n-1}a+\dfrac{n^2-n}{2}v^{n-2}a^2$, &c. So that when n is a whole number, it is evident the series will terminate, and the fluent be had in finite terms. Suppose $n=2$, and $m=3$, then the fluxion will become $v^{\frac{7}{3}}\dot{v}-2av^{\frac{4}{3}}\dot{v}+a^2v^{\frac{1}{3}}\dot{v}$, and the fluent by the common rule

is

is $\frac{3}{10}v^{\frac{10}{3}} - \frac{6}{7}av^{\frac{7}{3}} + \frac{3}{4}a^2v^{\frac{4}{3}}$; which, by restoring the value of v, becomes $\frac{3}{10}.\overline{a+x}|^{\frac{10}{3}} + \frac{3a^2}{4}.\overline{a+x}|^{\frac{4}{3}} - \frac{6a}{7}.\overline{a+x}|^{\frac{7}{3}}$, the required fluent.

And when the flowing part of the quantity under the radical is a power, it always follows by substitution, that if the exponent of the flowing quantity without the radical be an even number, the fluent may be had in finite terms; but if an odd number, it will run into an infinite series, because the index of the binomial then becomes a fraction.

Example.

Required the fluent of $x^n\dot{x}\sqrt[m]{a^2+x^2}$.

Put $a^2+x^2=v$, then will $x=\sqrt{v-a^2}$, $x^n=\overline{v-a^2}|^{\frac{n}{2}}$, and $\dot{x}=\frac{\dot{v}}{2x}=\frac{\dot{v}}{2\sqrt{v-a^2}}$; $\therefore$ $x^n\dot{x}\sqrt[m]{a^2+x^2}=\frac{v^{\frac{1}{m}}\dot{v}\times\overline{v-a^2}|^{\frac{n}{2}}}{2\sqrt{v-a^2}}=\frac{1}{2}v^{\frac{1}{m}}\dot{v}\times\overline{v-a^2}|^{\frac{n-1}{2}}$. From whence it appears, that if n be an odd number, the index will be an integer, and the series will terminate; but if it be an even number, the index will be a fraction, and the series will run on *sine fine*. Suppose $n=5$, and $m=2$, then the expression becomes $\frac{1}{2}v^{\frac{1}{2}}\dot{v}\times\overline{v-a^2}|^2$; which being expanded, the fluent of each term taken by the common rule, and the value of v restored, gives $\frac{1}{7}.\overline{a^2+x^2}|^{\frac{7}{2}}+\frac{1}{3}.\overline{a^2+x^2}|^{\frac{3}{2}}-\frac{2}{5}a^2.\overline{a^2+x^2}|^{\frac{5}{2}}$ for the required fluent.

If

If the flowing part of the quantity under the radical be raised to some power, and the fluxion without it be only that of the root, the fluent must be obtained by infinite series. This, indeed, may be often avoided; for if the given fluxion can be reduced to a form similar to that of the arch of a circle, then the length of that arch, which is always expressed in terms of either the sine, versed sine, tangent, or secant, and the radius, will be the required fluent.

Example.

Required the fluent of $\frac{c^2\dot{y}}{\sqrt{4c^2-4y^2}}$.

The expression $\frac{c^2\dot{y}}{\sqrt{4c^2-4y^2}}$ is $=\frac{cc\dot{y}}{2\sqrt{cc-yy}}=\frac{1}{2}c\times\frac{c\dot{y}}{\sqrt{cc-yy}}$; but $\frac{c\dot{y}}{\sqrt{cc-yy}}$ appears to be the fluxion of the arch of a circle, of which the radius is c, and sine y; and therefore when the degrees in the arch are known, the length of the arch is known also, which being multiplied by $\frac{1}{2}c$ gives the fluent of $\frac{c^2\dot{y}}{\sqrt{4c^2-4y^2}}$.

And when it happens that we cannot reduce the fluxion to any of those forms, yet, if we can by any reduction discover, that the numerator of the expression is the fluxion of the denominator (which frequently occurs in the solutions of problems) then will the hyperbolic logarithm of the denominator be the fluent required.

Example.

Required the fluent of $\frac{c^3\dot{x}}{c^2-x^2}$.

The expression $\frac{c^3\dot{x}}{c^2-x^2}$ is $=\frac{1}{2}c^2\times\frac{2c\dot{x}}{c^2-x^2}$; and $\frac{2c\dot{x}}{c^2-x^2}$ is $=\frac{2c\dot{x}}{\overline{c+x}\times\overline{c-x}}=\frac{2c\dot{x}}{\overline{c-x}|^2}\times\frac{c-x}{c+x}$; but this expression is the same as $\frac{2c\dot{x}}{\overline{c-x}|^2}\div\frac{c+x}{c-x}$, or $\frac{2c\dot{x}\times\overline{c-x}|^{-2}}{\overline{c+x}\times\overline{c-x}|^{-1}}$, where the numerator is evidently the fluxion of the denominator; therefore the hyp. log. of $\frac{c+x}{c-x}$ being multiplied by $\frac{1}{2}c^2$ will be the required fluent.

But as it would be rather foreign to our present purpose, and swell the book beyond its intended size, to endeavour to explain all the various methods of finding fluents by a proper number of examples, we shall therefore only give one or two more of reducing surd fluxionary expressions by infinite series, and Emerson's tables; which will serve as a specimen of the method of operation with all other similar forms. Suppose we select one from Emerson's Trigonometry p. 27, second Edit. where he gives $\frac{r\dot{y}}{\sqrt{rr-yy}}$ for the fluxion of the arch of a circle.

First Method.

$\frac{r\dot{y}}{\sqrt{rr-yy}}=\frac{\sqrt{r^2-y^2}\times r\dot{y}}{r^2-y^2}$, ∴ therefore the reduction will stand as follows,

$$\sqrt{r^2 - y^2}\left(r - \frac{y^2}{2r} - \frac{y^4}{8r^3} - \frac{y^6}{16r^5} - \frac{5y^8}{128r^7}, \text{ \&c.}\right.$$

$$\begin{array}{rl}
 & \underline{r^2} \\
2r - \frac{y^2}{2r}\Big) & -y^2 \\
 & \underline{-y^2 + \frac{y^4}{4r^2}} \\
2r - \frac{y^2}{r} - \frac{y^4}{8r^3}\Big) & -\frac{y^4}{4r^2} \\
 & \underline{-\frac{y^4}{4r^2} + \frac{y^6}{8r^4} + \frac{y^8}{64r^6}} \\
2r - \frac{y^2}{r} - \text{\&c.}\Big) & -\frac{y^6}{8r^4} - \frac{y^8}{64r^6} \\
 & \underline{-\frac{y^6}{8r^4} + \frac{y^8}{16r^6} + \text{\&c.}} \\
2r - \text{\&c.}\Big) & -\frac{5y^8}{64r^6} - \text{\&c.} \\
 & \underline{-\frac{5y^8}{64r^6} + \text{\&c.}}
\end{array}$$

And the root multiplied by $r\dot{y}$ gives

$$r^2\dot{y} - \frac{y^2\dot{y}}{2} - \frac{y^4\dot{y}}{8r^2} - \frac{y^6\dot{y}}{16r^4} - \frac{5y^8\dot{y}}{128r^6}, \text{ \&c.}$$

this divided by $r^2 - y^2$ will be in this form, *viz.*

$r^2 - y^2$

$$r^2 - y^2)\,r^2\dot{y} - \frac{y^2\dot{y}}{2} - \frac{y^4\dot{y}}{8r^2} - \frac{y^6\dot{y}}{16r^4} - \frac{5y^8\dot{y}}{128r^6}\ \&c.\left(\dot{y} + \frac{y^2\dot{y}}{2r^2} + \frac{3y^4\dot{y}}{2.4r^4} + \frac{5y^6\dot{y}}{2.4.2r^6} + \frac{7.5y^8\dot{y}}{2.4.2.8r^6}\right),\ \&c.$$

$$\underline{r^2\dot{y} - y^2\dot{y}}$$

$$+\frac{y^2\dot{y}}{2} - \frac{y^4\dot{y}}{8r^2}$$

$$\underline{+\frac{y^2\dot{y}}{2} - \frac{y^4\dot{y}}{2r^2}}$$

$$+\frac{3y^4\dot{y}}{8r^2} - \frac{y^6\dot{y}}{16r^4}$$

$$\underline{+\frac{3y^4\dot{y}}{8r^2} - \frac{3y^6\dot{y}}{2.4r^4}}$$

$$+\frac{5y^6\dot{y}}{2.4.2r^4} - \frac{5y^8\dot{y}}{128r^6}$$

$$\underline{+\frac{5y^6\dot{y}}{2.4.2r^4} - \frac{5y^8\dot{y}}{2.4.2r^6}}$$

$$+\frac{7.5y^8\dot{y}}{2.4.2.8r^6} - \&c.$$

$$\underline{+\frac{7.5y^8\dot{y}}{2.4.2.8r^6} - \&c.}$$

or. $\frac{r\dot{y}}{\sqrt{r^2-y^2}} = \dot{y} + \frac{y^2\dot{y}}{2r^2} + \frac{3y^4\dot{y}}{2.4r^4} + \frac{5y^6\dot{y}}{2.4.2r^6} + \frac{7.5y^8\dot{y}}{2.4.2.8r^8}$, &c.

and the fluent is

$$y + \frac{y^3}{3.2r^2} + \frac{3y^5}{5.2.4r^4} + \frac{5y^7}{7.2.4.2r^6} + \frac{5.7y^9}{9.2.4.2.8r^8}, \text{ \&c.}$$

or, $$y + \frac{y^3}{3.2r^2} + \frac{3y^5}{5.2.4r^4} + \frac{3.5y^7}{7.2.4.6r^6} + \frac{3.5.7y^9}{9.2.4.6.8r^8}, \text{ \&c.}$$

Second Method.

$$\frac{r\dot{y}}{\sqrt{r^2-y^2}} = \sqrt{r^2-y^2} \times r\dot{y} \times \frac{1}{r^2-y^2}; \text{ that is,}$$

$$r^2\dot{y} - \frac{y^2\dot{y}}{2} - \frac{y^4\dot{y}}{8r^2} - \frac{y^6\dot{y}}{16r^4} - \frac{5y^8\dot{y}}{128r^6}, \text{ \&c. } \times : \frac{1}{r^2} + \frac{y^2}{r^4} + \frac{y^4}{r^6} + \frac{y^6}{r^8}, \text{ \&c.}$$

Multiply each term of the first series into all the terms of the latter, rejecting those which exceed the power you intend carrying it to, and it will stand thus,

$$\begin{array}{llllll}
\dot{y} & + \frac{y^2\dot{y}}{r^2} & + \frac{y^4\dot{y}}{r^4} & + \frac{y^6\dot{y}}{r^6} & + \frac{y^8\dot{y}}{r^8}, & \text{\&c.} \\
 & - \frac{y^2\dot{y}}{2r^2} & - \frac{y^4\dot{y}}{2r^4} & - \frac{y^6\dot{y}}{2r^6} & - \frac{y^8\dot{y}}{2r^8}, & \text{\&c.} \\
 & & - \frac{y^4\dot{y}}{8r^4} & - \frac{y^6\dot{y}}{8r^6} & - \frac{y^8\dot{y}}{8r^8}, & \text{\&c.} \\
 & & & - \frac{y^6\dot{y}}{16r^6} & - \frac{y^8\dot{y}}{16r^8}, & \text{\&c.} \\
 & & & & - \frac{5y^8\dot{y}}{128r^8}, & \text{\&c.} \\
\hline
\dot{y} & + \frac{y^2\dot{y}}{2r^2} & + \frac{3y^4\dot{y}}{2.4r^4} & + \frac{5y^6\dot{y}}{2.4.2r^6} & + \frac{7.5y^8\dot{y}}{2.4.2.8r^8}, & \text{\&c.}
\end{array}$$

from whence the fluent is had as before.

Third

Third Method.

$$\frac{r\dot{y}}{\sqrt{r^2 - y^2}} = r \times \overline{r^2 - y^2}|^{-\frac{1}{2}} \times \dot{y},$$

By Form 16, in Emerson's excellent Treatise of Fluxions, we have $a = r^2$, $\beta = -1$, $z = y$, $n = 2$, $\frac{\mu}{\nu} = -\frac{1}{2}$; (or $\mu = -1$, $\nu = 2$,) and $\pi = 0$; then by substitution

$$\frac{a^{\frac{\mu}{\nu}} z^{\pi+1}}{\pi + 1} = \frac{r^{2 \times -\frac{1}{2}} y^{0+1}}{0 + 1} = \frac{r^{-1} y}{1} = \frac{y}{r},$$

$A = r^{-1} y$, $q = -\frac{y^2}{r^2}$, $\therefore$

$$\frac{\frac{\mu}{\nu} Aq}{\pi + 1 + n} = \frac{-\frac{1}{2} \times \frac{y}{r} \times -\frac{y^2}{r^2}}{0 + 1 + 2} = \frac{y^3}{3 \cdot 2r^3},$$

$B = + \frac{y^3}{2r^3}$, $\therefore$

$$\frac{\frac{\mu - \nu}{2\nu} Bq}{\pi + 1 + 2\nu} = \frac{\frac{-1-2}{4} \times \frac{y^3}{2r^3} \times -\frac{y^2}{r^2}}{0 + 1 + 4} = \frac{3y^5}{5 \cdot 2 \cdot 4r^5},$$

$C = + \frac{3y^5}{2 \cdot 4r^5}$, $\therefore$

$$\frac{\frac{\mu - 2\nu}{3\nu} Cq}{\pi + 1 + 3\nu} = \frac{\frac{-1-4}{6} \times \frac{3y^5}{2 \cdot 4r^5} \times -\frac{y^2}{r^2}}{0 + 1 + 6} = \frac{3 \cdot 5y^7}{7 \cdot 2 \cdot 4 \cdot 6r^7}, \text{ \&c.}$$

Therefore the formula gives

$$\frac{y}{r} + \frac{y^3}{3 \cdot 2r^3} + \frac{3y^5}{5 \cdot 2 \cdot 4r^5} + \frac{3 \cdot 5y^7}{7 \cdot 2 \cdot 4 \cdot 6r^7}, \text{ \&c.}$$

for the fluent of $\frac{\dot{y}}{\sqrt{r^2 - y^2}}$; which being multiplied by r, we have $y + \frac{y^3}{3 \cdot 2r^2} + \frac{3y^5}{5 \cdot 2 \cdot 4r^4}$, &c. as before.

It

It alſo appears, from the conſtruction of the fluxion, that the fluent will be expreſſed by the arch of a circle, of which the rad. is unity, and ſine $\frac{y}{r}$; which may be very conveniently had from the Tables of Nat. Sines, &c. thus; Find the degrees, &c. corresponding to the given value of $\frac{y}{r}$, which call d; then it will be, as $180 : 3{\cdot}14159 :: d : \frac{3{\cdot}14159d}{180}$ the required fluent.

This one example indeed might be ſufficient to ſhew the method of procedure with all ſuch expreſſions; but, in order to make it as plain as poſſible, we ſhall take another example, which is more complex, from *p.* 30 of the ſame book, *viz.* $\frac{r\dot{v}}{\sqrt{2rv-vv}}$; being the fluxion of the arch of a circle, expreſſed in terms of the radius and verſed ſine.

$$\frac{r\dot{v}}{\sqrt{2rv-vv}} = \frac{\sqrt{2rv-vv}}{2rv-vv} \times r\dot{v}, \therefore$$

$2rv-v^2$

$$\frac{2rv-v^2}{2rv}\left(\sqrt{2rv}-\frac{v^2}{2\sqrt{2rv}}-\frac{v^4}{2\cdot 4\sqrt{2rv}\,|^3}-\frac{v^6}{2\cdot 2\cdot 4\sqrt{2rv}\,|^5},\ \&c.\right.$$

$$\left.2\sqrt{2rv}-\frac{v^2}{2\sqrt{2rv}}\right)\quad -v^2$$

$$-v^2+\frac{v^4}{4\sqrt{2rv}\,|^2}$$

$$\left.2\sqrt{2rv}-\frac{v^2}{\sqrt{2rv}},\ \&c.\right)\quad -\frac{v^4}{4\cdot 2rv}$$

$$-\frac{v^4}{4\cdot 2rv}+\frac{v^6}{2\cdot 4\sqrt{2rv}\,|^4},\ \&c.$$

$$\left.2\sqrt{2rv},\ \&c.\right)\quad -\frac{v^6}{2\cdot 4\cdot 2rv\,|^2},\ \&c.$$

$$-\frac{v^6}{2\cdot 4\cdot 2rv\,|^2},\ \&c.$$

2rv—vv

$$2rv-vv)\sqrt{2rv}-\frac{v^3}{2\sqrt{2rv}}-\frac{v^4}{2.4\overline{\sqrt{2rv}}|^3}-\frac{v^6}{4.4\,\overline{\sqrt{2rv}}|^5},\ \&c.\left(\frac{\sqrt{2rv}}{2rv}+\frac{\sqrt{2rv}.v^2}{2.\overline{2rv}|^2}+\frac{3.\sqrt{2rv}.v^4}{4.2.\overline{2rv}|^3}+\frac{15.\sqrt{2rv}.v^6}{6.4.2.\overline{2rv}|^4},\ \&c.\right.$$

$$\sqrt{2rv}-\frac{v\sqrt{2rv}}{2r}$$

$$+\frac{v^2\sqrt{2rv}}{2.2rv}-\frac{v^4}{2.4\overline{\sqrt{2rv}}|^3}$$

$$+\frac{v^2\sqrt{2rv}}{2.2rv}-\frac{\sqrt{2rv}.v^4}{2.\overline{2rv}|^2}$$

$$+\frac{3.\sqrt{2rv}.v^4}{4.2.\overline{2rv}|^2}-\frac{v^6}{4.4\overline{\sqrt{2rv}}|^5},\ \&c.$$

$$+\frac{3\sqrt{2rv}.v^4}{4.2.\overline{2rv}|^2}-\frac{3.\sqrt{2rv}.v^6}{4.2.\overline{2rv}|^3},\ \&c.$$

$$+\frac{3.5\sqrt{2rv}.v^6}{6.4.2.\overline{2rv}|^3},\ \&c.$$

$$+\frac{3.5\sqrt{2rv}.v^6}{6.4.2.\overline{2rv}|^3},\ \&c.$$

The

The quotient multiplied by $r\dot{v}$ gives

$$\frac{\sqrt{2rv}\; r\dot{v}}{2rv} + \frac{\sqrt{2rv}.v^2 r\dot{v}}{2.\overline{2rv}|^2} + \frac{3\sqrt{2rv}.v^4 r\dot{v}}{4.2.\overline{2rv}|^3} + \frac{3.5\sqrt{2rv}.v^6 r\dot{v}}{6.4.2.\overline{2rv}|^4}, \text{ \&c.}$$

$$\text{or, } \tfrac{1}{2}\sqrt{2r}.v^{-\frac{1}{2}}\dot{v} + \tfrac{1}{4} \times \frac{\sqrt{2r}.v^{\frac{1}{2}}\dot{v}}{2^2 r} + \frac{3}{4.2} \times \frac{\sqrt{2r}.v^{\frac{3}{2}}\dot{v}}{2^3.r^2} + \frac{3.5}{6.4.2} \times \frac{\sqrt{2r}.v^{\frac{1}{2}}v^2\dot{v}}{2^4.r^3}, \text{ \&c.}$$

And the fluent is,

$$\sqrt{2rv} + \frac{\sqrt{2rv}.v}{2^2.3r} + \frac{3.\sqrt{2rv}.v^2}{2^3.4.5r^2} + \frac{3.5\sqrt{2rv}.v^3}{2^4.4.6.7r^3}, \text{ \&c.}$$

$$\text{or, } \sqrt{2rv} \times : 1 + \frac{v}{2^2.3r} + \frac{3v^2}{2^3.4.5r^2} + \frac{3.5v^3}{2^4.4.6.7r^3}, \text{ \&c.}$$

The fluent may also be had by Form 16, Emerson's Fluxions, as follows:

$\frac{r\dot{v}}{\sqrt{2rv-vv}} = r \times \overline{2r-v}|^{-\frac{1}{2}} \times v^{-\frac{1}{2}}\dot{v}$, therefore, $a = 2r$, $\beta = -1$, $z = v$, $n = 1$, $\pi = -\frac{1}{2}$, and $\frac{\mu}{\nu} = -\frac{1}{2}$; or $\mu = -1$, and $\nu = 2$.

And by substitution,

$$\frac{a^{\frac{\mu}{\nu}}z^{\pi+1}}{\pi+1} = \frac{\overline{2r}|^{-\frac{1}{2}}v^{\frac{1}{2}}}{\frac{1}{2}} = \frac{2\sqrt{v}}{\sqrt{2r}},$$

$$A = \frac{\sqrt{v}}{\sqrt{2r}};\quad q = -\frac{v}{2r}, \therefore$$

$$\frac{\frac{\mu}{\nu}Aq}{\pi+1+\nu} = \frac{-\frac{1}{2}\times\frac{\sqrt{v}}{\sqrt{2r}}\times-\frac{v}{2r}}{-\frac{1}{2}+1+1} = \frac{2v^{\frac{1}{2}}v}{3.2^2r\sqrt{2r}},$$

B =

$$B = \frac{vv^{\frac{1}{2}}}{2^2 r\sqrt{2r}},\ \therefore$$

$$\frac{\frac{\mu-\nu}{2\nu}\,Bq}{\pi+1+2\eta} = \frac{\frac{-1-2}{4}\times\frac{vv^{\frac{1}{2}}}{2^2r\sqrt{2r}}\times-\frac{v}{2r}}{\frac{5}{2}} = \frac{2.3v^2v^{\frac{1}{2}}}{\sqrt{2r}.4.2^3.5r^2},$$

$$C = \frac{3v^2v^{\frac{1}{2}}}{4.2^3r^2\sqrt{2r}},\ \therefore$$

$$\frac{\frac{\mu-2\nu}{3\nu}\,Cq}{\pi\times1\times3\nu} = \frac{\frac{-1-4}{6}\times\frac{3v^2v^{\frac{1}{2}}}{4.2^3r^2\sqrt{2r}}\times-\frac{v}{2r}}{-\frac{1}{2}+1+3} = \frac{2.3.5v^3v^{\frac{1}{2}}}{2^4.4.6.7r^3\sqrt{2r}}.$$

&c.

Hence we have

$$\frac{2\sqrt{v}}{\sqrt{2r}} + \frac{2v^{\frac{1}{2}}v}{3.2^2r\sqrt{2r}} + \frac{2.3v^2v^{\frac{1}{2}}}{5.4.2^3r^2\sqrt{2r}} + \frac{2.3.5v^3v^{\frac{1}{2}}}{7.4.6.2^4r^3\sqrt{2r}},\ \&c.$$

for the fluent of $\frac{\dot{v}}{\sqrt{2rv-vv}}$. Multiply by r, and it becomes

$$\frac{2rv^{\frac{1}{2}}}{\sqrt{2r}} + \frac{2rv^{\frac{1}{2}}v}{2^2.3r\sqrt{2r}} + \frac{3.2rv^{\frac{1}{2}}v^2}{2^3.4.5r^2\sqrt{2r}} + \frac{3.5.2rv^{\frac{1}{2}}v^3}{2^4.4.6.7r^3\sqrt{2r}},\ \&c.$$

or, $\sqrt{2rv} + \frac{\sqrt{2rv}.v}{2^2.3r} + \frac{\sqrt{2rv}.3v^2}{2^3.4.5r^2} + \frac{\sqrt{2rv}.3.5.v^3}{2^4.4.6.7r^3}$, &c.

for the fluent of $\frac{r\dot{v}}{\sqrt{2rv-vv}}$, as before.

Or thus,

$\frac{r\dot{v}}{\sqrt{2rv-vv}} = \frac{rv^{-\frac{1}{2}}\dot{v}}{\sqrt{2r-v}}$. This correſponds to Form 10, where $z = v$, $a = 2r$, $-\beta = -1$, (or $\beta = 1$,) and $\eta = 1$; and the fluent is, $2Nr \times$ Degrees of the arch of a circle, of

 which

which the radius is 1, and natural sine $\sqrt{\frac{v}{2r}}$. Or, putting d for the degrees, it will be,

as $180 : 3{\cdot}14159 :: 2dr : \frac{3{\cdot}14159 \times 2dr}{180}$, the fluent.

As these examples will be sufficient to explain the method of obtaining fluents by infinite series, I shall therefore only farther observe to the learner, that, in the solutions of questions, the fluent first found generally wants correcting, by the addition or subtraction of some constant quantity; which is always to be determined by the nature of the question, and may be effected by this easy rule: Substitute for the variable quantity in the fluent first found, that particular value which it is known to have when the *whole* fluent is supposed to be equal nothing; then, if the resulting quantity be affirmative, subtract it from the fluent before found, but if negative add it; and it will be truly corrected. If the whole expression should vanish, the fluent needs no correction. An example or two will make it plain. Thus, in the fluxionary expression $\dot{y} \times \overline{a+y}|^4$, the fluent first found is $\frac{\overline{a+y}|^5}{5}$. Now suppose when the *whole* fluent which this ought to express is equal to nothing, that y is also equal to nothing, then will $\frac{\overline{a+y}|^5}{5}$ become $+\frac{a^5}{5}$; hence $\frac{\overline{a+y}|^5}{5}$ exceeds the whole fluent by $\frac{a^5}{5}$, which being therefore subtracted from it, leaves $\frac{\overline{a+y}|^5 - a^5}{5}$ for the correct fluent. For if we put z for the whole fluent, then ought this equation $z = \frac{\overline{a+y}|^5}{5}$ to hold good in all the cotemporary values of z and y. But it appears that when $z = 0$, if we suppose $y = 0$ also, the expression for the fluent, instead

instead of vanishing, remains equal to $\frac{a^5}{5}$; this quantity therefore must evidently be either subtracted from $\frac{\overline{a+y}|^5}{5}$, or added to x to preserve the equality. From whence the rule is obvious.

If when the *whole* fluent is supposed equal to nothing, y should be known, from the nature of the problem, to have a certain value, still the correction will be performed in the same manner. Thus, when the *whole* fluent is equal nothing, let $y = b$; then the above fluent corrected will be $\frac{\overline{a+y}|^5 - \overline{a+b}|^5}{5}$. As this is a very material point in the use of fluxions, it may not be amiss to illustrate it by the solutions of a few questions, where the fluents will require a correction.

1. Required the area of a curve, of which the equation is $x^4 + a^2y^2 = a^2x^2$.

By reduction we have $y = \frac{x \times \sqrt{a^2 - x^2}}{a}$; multiply both sides by the fluxion of the abcifs, and we get $\frac{\overline{a^2 - x^2}|^{\frac{1}{2}} \times x\dot{x}}{a}$ $= y\dot{x} =$ the fluxion of the area; the fluent of which found by the common rule, is $- \frac{\overline{a^2 - x^2}|^{\frac{3}{2}}}{3a}$. Now when the area, or whole fluent, is supposed to be $= 0$, it is evident that the abcifs x must be also $= 0$; hence the above expression becomes $- \frac{a^2}{3}$, and therefore $\frac{a^2}{3} - \frac{\overline{a^2 - x^2}|^{\frac{3}{2}}}{3a}$ will be the correct fluent, or true value of the area.

2. Let the equation of a curve be $x^2y^2 + a^2y^2 = a^4$, required the area.

By reducing the equation we get $y = \frac{a^2}{\sqrt{a^2+x^2}}$; hence $\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}$ is the fluxion of the area. And because $\frac{\dot{x}}{\sqrt{a^2+x^2}}$

is $= \frac{\frac{\dot{x}}{\sqrt{a^2+x^2}} \times \overline{\sqrt{a^2+x^2}+x}}{\sqrt{a^2+x^2}+x}$, where the numerator is evidently the fluxion of the denominator, therefore the fluent is $a^2 \times$ hyp. log. of $x + \sqrt{a^2+x^2}$. Now put $x = 0$, then the fluent will become $a^2 \times$ hyp. log. of a, $\therefore a^2 \times$ hyp. log. of $\overline{x + \sqrt{a^2+x^2}} - a^2 \times$ hyp. log. of $a = a^2 \times$ hyp. log. of $\frac{x+\sqrt{a^2+x^2}}{a}$ is the correct value of the area.

3. To find the superficies of an hyperbolic Conoid.

Put the semitransverse of the generating hyperbola $= t$, the semiconjugate $= c$, and the distance of any ordinate from the center $= x$; then by the property of the curve we have $y = \frac{c}{t}\sqrt{x^2-t^2}$, and therefore $\dot{y} = \frac{cx\dot{x}}{t\sqrt{x^2-t^2}}$; hence the fluxion of the area will be (putting $3{\cdot}141592 = p$)

$\frac{2pc\dot{x}}{t^2} \times \sqrt{\overline{t^2+c^2} \times x^2 - t^4}$; or $\left(\text{by putting } a^2 = \frac{t^4}{t^2+c^2}\right)$

$\frac{2pc\dot{x}}{a}\sqrt{x^2-a^2}$. Now since the fluxion without the radical is not in any given ratio to that of the quantity under it, but is the fluxion of the root only, therefore, in order to avoid an infinite series for the fluent, let the variable part of the fluxion be changed to $\frac{\dot{x} \times \overline{x^2-a^2}}{\sqrt{x^2-a^2}}$, which is equal to $\frac{x^3\dot{x} - \frac{1}{2}a^2x\dot{x}}{\sqrt{x^4-a^2x^2}}$

$- \frac{\frac{1}{2}a^2x\dot{x}}{\sqrt{x^4-a^2x^2}}$. Here then the numerator of the first term

is

is in a given ratio to the fluxion of the quantity under the radical, whence we get $\frac{pcx\sqrt{x^2-a^2}}{a}$ for the fluent of that term; from which subtracting the fluent of the other term $= \frac{\frac{1}{2}a^2\dot{x}}{\sqrt{x^2-a^2}}$, there arises $\frac{pcx\sqrt{x^2-a^2}}{a} - pca \times$ hyp. log. of $x + \sqrt{x^2-a^2}$. But from the nature of the problem it is evident, that when the area is supposed to be $= 0$, that x will then be $= a$; therefore the correct fluent will be

$$\frac{pcx}{a}\sqrt{x^2-a^2} - pc^2 - pcd \times \text{hyp. log. of } \frac{x+\sqrt{x^2-a^2}}{t+\frac{ac}{t}} =$$

the true area of the superficies.

Before we close this subject, it may not be amiss to give a short Explanation of Trigonometrical Fluxions. These should by all means be well understood by the young Mathematician, on account of their great usefulness in Astronomy, Navigation, Dialling, &c. For since the places of the heavenly bodies, the times of their rising, setting, &c. their right ascension, declination, latitude, longitude, amplitude, azimuth, &c. are all calculated by spherical triangles, of which the sides and angles are variously affected by parallax, refraction, precession of the equinoxes, obliquity of the ecliptic, &c. it is obvious, that where accuracy is required, it is necessary to make a correction of the variable parts, by supposing a small continued motion in one or more of the great circles by which the triangle is formed, and thence by the fluxionary increase or decrease of the sides and angles to determine the quantities sought.

This subject naturally divides itself into four cases as follow:

 1. When

1. When an angle and a ſide adjacent are conſtant. 2. When an angle and the ſide oppoſite are conſtant. 3. When two of the ſides are conſtant. 4. When two of the angles are conſtant.

CASE I. Fig. 14.

In any ſpherical triangle ABC, if the angle B and a ſide adjacent to this angle, ſuppoſe BA, be conſtant or invariable, then ſhall we have the following analogies,

1. $\dot{\overline{BC}} : \dot{\overline{AC}} :: R : \text{coſ.}\,C.$
2. $\dot{\overline{BC}} : \dot{A} :: \text{ſin.}\,AC : \text{ſin.}\,C.$
3. $\dot{\overline{AC}} : \dot{A} :: \text{ſin.}\,AC \times \text{coſ.}\,C : R \times \text{ſin.}\,C :: \text{ſin.}\,AC : \text{tang.}\,A.$
4. $\dot{\overline{BC}} : \dot{C} :: \text{tang.}\,AC : \text{ſin.}\,C.$

That is,

1. As the fluxion of the variable ſide adjacent to the conſtant angle is to the fluxion of the ſide oppoſite, ſo is radius to the coſine of the angle oppoſite to the conſtant ſide.

2. As the fluxion of the variable ſide adjacent to the conſtant angle is to the fluxion of the angle oppoſite to this ſide, ſo is the ſine of the ſide oppoſite to the conſtant angle to the ſine of the angle oppoſite to the conſtant ſide.

3. As the fluxion of the ſide oppoſite to the conſtant angle is to the fluxion of the variable angle adjacent to the conſtant ſide, ſo is the rectangle under the ſine of this ſide and the coſine of the third angle to the rectangle under the ſine of the ſame angle and the radius, ſo is the ſine of the ſide oppoſite to the conſtant angle to the tangent of the angle oppoſite to the conſtant ſide.

4. As

4. As the fluxion of the side adjacent to the constant angle is to the fluxion of the angle adjacent to this side, so is the tangent of the side opposite to the constant angle to the sine of the angle opposite to the constant side.

In order to give a clear demonstration of these analogies, it will be necessary to premise the following

LEMMA.

If from the three angles of any spherical triangle ABC (fig. 15.) taken as poles, there be formed another spherical triangle DEF, we shall have, $\dot{\overline{DE}} = \dot{A}$, $\dot{\overline{DF}} = \dot{C}$, $\dot{\overline{FE}} = \dot{A}$; and $\dot{D} = \dot{\overline{AC}}$, $\dot{E} = \dot{\overline{AB}}$, $\dot{F} = \dot{\overline{BC}}$. For it is demonstrated by most writers on spherics, that each side of the new triangle so described will be the supplement of the angle which is at its pole, and each of its angles the supplement of that side of the triangle ABC, to which it is opposite; and it is well known that an arc and its supplement have the same sine, cosine, tangent, &c. therefore the fluxions of the sides DE, EF, and FD, of the triangle DEF will be respectively equal to the fluxions of the opposite angles in the triangle ABC; and the fluxions of the angles D, E, F, the same as those of the opposite sides AC, AB, BC.

Suppose now in the triangle ABC that the great circle of which AC is a part, by a motion round A as a pole, comes into the position A*C*, and the sides BC, AC will become B*c*, C*c*, respectively, which produce till they be each 90 degrees, and from the pole A describe the little circular arcs C*C*, E*E*; then will the indefinitely small quantities C*c*, *Cc*, be the respective fluxions of the sides BC, AC, and the arc E*E* the measure of the variation of the angle A; and because the

 triangle

triangle CCc, which is right-angled at C, may be eſteemed right-lined on account of its ſmallneſs, the ſides will be proportional to the ſines of their oppoſite angles. But as the angle at c is eſſentially the ſame as the angle at C, and conſequently the angle cCC the complement of that angle, we ſhall evidently have, $Cc : Cc :: R : \text{coſ.} C$; or, $\dot{\overline{BC}} : \dot{\overline{AC}} :: R : \text{coſ.} C$, the firſt analogy. 2. Becauſe the ſmall arcs EE, CC, contain the ſame number of degrees, it will be, $EE : CC :: R : \text{ſin.} AC$; and in the triangle CCc, it is, $CC : Cc :: \text{ſin.} C : R$, hence $EE \times CC : Cc \times CC :: R \times \text{ſin.} C : R \times AC$, that is, $EE : Cc :: \text{ſin.} C : \text{ſin.} AC$; or, $\dot{A} : \dot{\overline{BC}} :: \text{ſin.} C : \text{ſin.} AC$, and by inverſion, $\dot{\overline{BC}} : \dot{A} :: \text{ſin.} AC : \text{ſin.} C$, the ſecond analogy. 3. By inverſion, and multiplying the correſponding terms of the firſt and ſecond analogies, we get,

$\dot{\overline{AC}} : \dot{A} :: \text{ſin.} AC \times \text{coſ.} C : \text{ſin.} C \times R :: \text{ſin.} AC : \text{tang.} C$, the third analogies. 4. And according to this caſe in the triangle DEF (fig. 15.) of which the parts are all ſupplements to thoſe of the triangle ABC, the angle at E and the adjacent ſide FE will be conſtant; and by the Lemma, the fluxions of the ſide DF and angle F will be reſpectively equal to the fluxions of the angle C and ſide BC of the triangle ABC; but by the laſt analogy it will be, $\dot{\overline{DF}} : \dot{F} :: \text{ſin.} DF : \text{tang.} D$, and becauſe angles and their ſupplements have the ſame ſine and tangent, we have by ſubſtitution, $\dot{C} : \dot{\overline{BC}} :: \text{ſin.} C : \text{tang.} AC$, and by inverſion, $\dot{\overline{BC}} : \dot{C} :: \text{tang.} AC : \text{ſin.} C$, the fourth analogy. *Q. E. D.*

CASE II. Fig. 16.

When any one of the angles B and its oppoſite ſide AC are conſtant, we have theſe analogies,

1. $\dot{C}$

1. $\dot{C} : \dot{\overline{AB}} :: \text{tang.}\, C : \text{tang.}\, AB.$

2. $\dot{A} : \dot{\overline{BC}} :: \text{tang.}\, A : \text{tang.}\, BC.$

3. $\dot{\overline{AB}} : \dot{\overline{BC}} :: \text{cof.}\, C : \text{cof.}\, A.$

4. $\dot{A} : \dot{C} :: \text{cof.}\, BC : \text{cof.}\, AB.$

That is,

1, 2. As the fluxion of either of the variable angles is to the fluxion of its oppofite fide, fo is the tangent of this angle to the tangent of the fame fide.

3, 4. The fluxions of the variable fides are as the cofines of their oppofite angles; and the fluxions of the variable angles as the cofines of their oppofite fides.

DEMONSTRATION.

Since the fin. AC : fin. B :: fin. AB : fin. C :: fin. BC : fin. A, we have, $\text{fin.}\, C = \frac{\text{fin.}\, B}{\text{fin.}\, AC} \times \text{fin.}\, AB$, and $\text{fin.}\, A = \frac{\text{fin.}\, B}{\text{fin.}\, AC} \times \text{fin.}\, BC$; from whence it appears that the fluxions of fin. C, and fin. A, are as the fluxions of their oppofite fides AB and BC; and the fluxion of the arc which meafures the angle C is to the fluxion of the arc AB as $\frac{\dot{\overline{\text{fin.}\, C}}}{\text{cof.}\, C}$ is to $\frac{\dot{\overline{\text{fin.}\, AB}}}{\text{cof.}\, AB}$*, or as $\dot{C} : \dot{\overline{AB}} :: \frac{\dot{\overline{\text{fin.}\, C}}}{\text{cof.}\, C} : \frac{\dot{\overline{\text{fin.}\, AB}}}{\text{cof.}\, AB}$. And becaufe the fin. C has a conftant ratio to fin. AB, it will be,

* For it is demonftrated, in moft treatifes on fluxions, that the fluxion of an arc is to that of its fine as radius is to the cofine of this arc; and therefore when the radius is 1 we have $\frac{\dot{\overline{\text{fine}}}}{\text{cof.}}$ for the fluxion of the correfponding arc.

as

as $\overline{\dot{\text{fin}}.\, C} : \overline{\dot{\text{fin}}.\, AB} :: \text{fin}.\, C : \text{fin}.\, AB$; hence by fuftitution, $\dot{C} : \overline{\dot{AB}} :: \frac{\overline{\dot{\text{fin}}.\, C}}{\text{cof}.\, C} : \frac{\overline{\dot{\text{fin}}.\, AB}}{\text{cof}.\, AB}$, that is $\dot{C} : \overline{\dot{AB}} :: \text{tang}.\, C : \text{tang}.\, AB$, the firft analogy. After the fame manner it may be proved that, $\dot{A} : \overline{\dot{BC}} :: \text{tang}.\, A : \text{tang}\ BC$, the fecond analogy. And if D*A* be fuppofed equal to DA, D*C* equal to DC, and the angle at D very fmall; the angles at *A* and *C* may evidently be efteemed right angles, *AC* equal to *ac*, and therefore *Aa* equal to *Cc*. Hence in the right-angled triangle A*Aa*, we have A*a* : *Aa* :: R : cof. A; and in the right-angled triangle C*Cc*, *Cc* : C*c* :: cof. C : R; therefore A*a* × *Cc* : C*c* × *Aa* :: R × cof. C : R × cof. A, that is A*a* : C*c* :: cof. C : cof. A; or, $\overline{\dot{AB}} : \overline{\dot{BC}} :: \text{cof}.\, \dot{C} : \text{cof}.\, A$, the third analogy. And by the fame manner of reafoning in the triangle DEF (fig. 15.) we have $\overline{\dot{DE}} : \overline{\dot{DF}} :: \text{cof}.\, F : \text{cof}.\, E$, therefore *per* lem. it will be $\dot{A} : \dot{C} :: \text{cof}.\, BC : \text{cof}.\, AB$, the fourth analogy. *Q. E. D.*

CASE. III. Fig. 17.

If any two of the fides AB, BC, are conftant, thefe analogies will obtain,

1. $\dot{B} : \dot{A} :: R \times \text{fin}.\, AC : \text{fin}.\, BC \times \text{cof}.\, C.$
2. $\dot{B} : \dot{C} :: R \times \text{fin}.\, AC : \text{fin}.\, AB \times \text{cof}.\, A.$
3. $\dot{B} : \overline{\dot{AC}} :: R^2 : \text{fin}.\, C \times \text{fin}.\, BC :: R^2 : \text{fin}.\, A \times \text{fin}.\, AB.$
4. $\dot{C} : \overline{\dot{AC}} :: \text{cot}.\, A : \text{fin}.\, AC.$
5. $\dot{A} : \overline{\dot{AC}} :: \text{cot}.\, C : \text{fin}.\, AC.$

That

That is,

1, 2. The fluxion of the angle formed by the two constant sides is to the fluxion of either of the other two angles, as the rectangle under the sine total and sine of the variable side is to the rectangle under the sine of the side opposite to the latter angle, and the cosine of the third angle adjacent to this side.

3. As the fluxion of the angle formed by the constant sides is to the fluxion of its opposite side, so is the square of the radius to the rectangle under the side of either of the other angles and the sine of its adjacent side.

4, 5. As the fluxion of either of the angles adjacent to the variable side is to the fluxion of this side, so is the cotangent of the other adjacent angle to the sine of the said side.

DEMONSTRATION.

Let the angle ABC become AB*C*, and the sides BC, B*C*, AC, A*C*, be produced to quadrants; then will E*e* evidently measure the variation of the angle at B, and G*g* that of the angle at A. From A describe C*c*, and there will be formed the triangle C*c*C, which on account of the smallness of the angle CA*C* may be esteemed right-angled at *c*, as also the angle *c*CC the complement of the angle C. From hence, and the similar sectors, we have these proportions,
E*e* : C*C* :: R : sin. BC; C*c* : G*g* :: sin. AC : R; C*C* : C*c* :: R : cos. C; the corresponding terms of which being multiplied, and the values of the arcs E*e*, G*g*, substituted in the product, we shall have
$\dot{B} : \dot{A} :: R \times$ sin. AC : sin. BC $\times$ cos. C, the first analogy. By a similar process we get, $\dot{B} : \dot{C} :: R \times$ sin. AC : sin. AB $\times$ cos. A, the second analogy. And by multiplying the corresponding terms

terms of these two proportions, $Ee : CC :: R : \text{sin.}\,BC$; $CC : cC :: R : \text{sin.}\,C$, we find $Ee : cC :: R^2 : \text{sin.}\,BC \times \text{sin.}\,C$, or, $\dot{B} : \dot{\overline{AC}} :: R^2 : \text{sin.}\,BC \times \text{sin.}\,C$; and if we suppose the angle B to flow towards A, we shall in like manner get $\dot{B} : \dot{\overline{AC}} :: R^2 : \text{sin.}\,AB \times \text{sin.}\,A$, which are the third analogies. By Case II. we have $\dot{B} : \dot{C} :: R \times \text{sin.}\,AC : \text{sin.}\,AB \times \text{cos.}\,A$, and $\dot{\overline{AC}} : \dot{B} :: \text{sin.}\,C \times \text{sin.}\,BC : R^2$; therefore, by multiplying these two proportions together and reducing the product, we have, $\dot{\overline{AC}} : \dot{C} :: \text{sin.}\,AC : \text{cot.}\,A$; or by inversion, $\dot{C} : \dot{\overline{AC}} :: \text{cot.}\,A : \text{sin.}\,AC$, the fourth analogy. And by the same method we find $\dot{A} : \dot{\overline{AC}} :: \text{cot.}\,C : \text{sin.}\,AC$, the fifth analogy. *Q. E. D.*

CASE IV. Fig. 16.

When any two of the angles B, C, are constant, the following analogies are derived,

1. $\dot{\overline{AC}} : \dot{\overline{AB}} :: \text{sin.}\,B \times R : \text{sin.}\,C \times \text{cos.}\,BC$.

2. $\dot{\overline{AC}} : \dot{\overline{BC}} :: \text{sin.}\,B \times R : \text{sin.}\,A \times \text{cos.}\,AB$.

3. $\dot{\overline{AC}} : \dot{B} :: \text{cosec.}\,AB : \text{sin.}\,A :: \text{cosec.}\,BC : \text{sin.}\,C$.

4. $\dot{\overline{AB}} : \dot{B} :: \text{cot.}\,BC : \text{sin.}\,B$.

5. $\dot{\overline{BC}} : \dot{B} :: \text{cot.}\,AB : \text{sin.}\,B$.

That is,

1, 2. As the fluxion of the side opposite to the variable angle is to the fluxion of either of the other sides, so is the rectangle under the radius and sine of the variable angle to the rectangle under the sine of the angle opposite to this other side and the cosine of the third side.

3. As

3. As the fluxion of the side opposite to the variable angle is to the fluxion of this angle, so is the cosecant of either of the other sides to the sine of the constant angle adjacent to this side.

4, 5. As the fluxion of either of the sides opposite to the constant angles is to the fluxion of the variable angle, so is the cotangent of the other side to the sine of the said angle.

DEMONSTRATION.

By applying this case to fig. 15. the two sides DF, DE, will be variable, therefore by the last case we shall have, $\dot{D} : \dot{E} :: R \times \text{sin. } FE : \text{sin. } DE \times \text{cos. } F$; or, by substituting the corresponding values in the triangle ABC,
$\dot{\overline{AC}} : \dot{\overline{AB}} :: R \times \text{sin. } B : \text{sin. } C \times \text{cos. } BC$, the first analogy. In like manner may the second analogy be obtained. And because the square of the radius divided by the sine of an arc is equal to the cosecant of this arc, the third analogies will become $\dot{\overline{AC}} : \dot{B} :: R^2 : \text{sin.}A \times \text{sin.}AB :: R^2 : \text{sin.}C \times \text{sin.}BC$, and these proportions are easily investigated by the triangle DEF, and the third analogy of Case III. The fourth and fifth analogies are immediately deduced by applying those of Case III. to Fig. 15. *Q.E.D.*

I shall now shew the application of this theory by an example or two.

Example 1.

Required the hour of the day or night, by the observed altitude of a star; and also the error of time, supposing the error in the observed altitude to be known.

Solution.

Solution. In the ſpherical triangle PSZ, (fig. 18.) wherein P repreſents the pole, Z the zenith of the place of obſervation, and S the place of the ſtar, ZS will be the complement of the ſtar's altitude, PS the complement of its declination; PZ the complement of the latitude, and the angle ZPS a variable hour-angle contained betwixt the conſtant ſides PS, PZ. Now by Caſe III. we get, $\dot{P} : \dot{\overline{ZS}} :: R^2 : \text{ſin. } Z \times \text{ſin. } PZ$; but $\dot{P}$ is evidently the meaſure of the error in time, therefore we have $\dot{P} = \dfrac{R^2 \times \dot{\overline{ZS}}}{\text{ſin. } Z \times \text{ſin. } PZ}$: *Q. E. D.*

Cor. 1. From this equation it appears that, ſince $\dfrac{R^2}{\text{ſin. } Z}$ is a conſtant quantity, if the ſame error be ſuppoſed to be made in different obſervations in the ſame latitude, the error of time will not be altered, whatever the altitude of the ſtar be.

2. If the latitude be ſtill ſuppoſed the ſame, the error of time will be the leaſt when the ſtar is obſerved on the prime vertical; for then the angle at Z is a right angle, and therefore $\dfrac{R^2}{\text{ſin } Z}$ becomes barely R. And the error will be the leaſt poſſible if the obſervation be made at the equator when the ſtar is on the prime vertical, for then the expreſſion $\dfrac{R^2 \times \dot{\overline{ZS}}}{\text{ſin.} Z \times \text{ſin.} PZ}$ will evidently be a minimum, ſince ſin. Z × ſin. PZ will be the greateſt poſſible.

Example II.

Required the correction neceſſary to be made in determining the inſtant the ſun comes to the meridian from two equal altitudes; ſuppoſing the ſun's declination to undergo a little change during the interval of the obſervations.

Solution.

Solution. If the ſun's declination was the ſame at both obſervations, half the interval would aſcertain the inſtant of noon; but as this can happen only at the ſolſtices, at all other times of the year this half interval muſt want a correction, by ſomething being ſubtracted when the ſun is in the aſcending ſigns, and added when in the deſcending ſigns; becauſe in the former caſe the ſun muſt evidently arrive later at the ſame altitude, and in the latter caſe ſooner. In order then to determine this correction, let P*s*, PS (fig. 19.) be two-hour circles very near to each other, and terminating in the ſame almacanter ESL, and the angle PS*s* will evidently expreſs how much later or ſooner the ſun comes to the ſame almacanter in the afternoon than in the morning; and therefore the half of it will expreſs the time which muſt be ſubtracted from, or added to, the half interval of the obſervations. Now as the change in declination for ſo ſmall an interval as three or four hours, may be ſafely taken for the naſcent variation or fluxion of the arc PS, and that the ſides PZ, SZ, of the triangle PSZ are conſtant, and the ſide PS variable; we ſhall have by Caſe III. as $\dot{P} : \dot{\overline{PS}}$ :: cot. S : ſin. PS; or, (*per* ſpherics)

$$\dot{P} : \dot{\overline{PS}} :: \frac{\text{cot. PZ}}{\text{ſin. P}} \pm \frac{\text{cot. PS}}{\text{tang. P}} : R; \text{ hence}$$

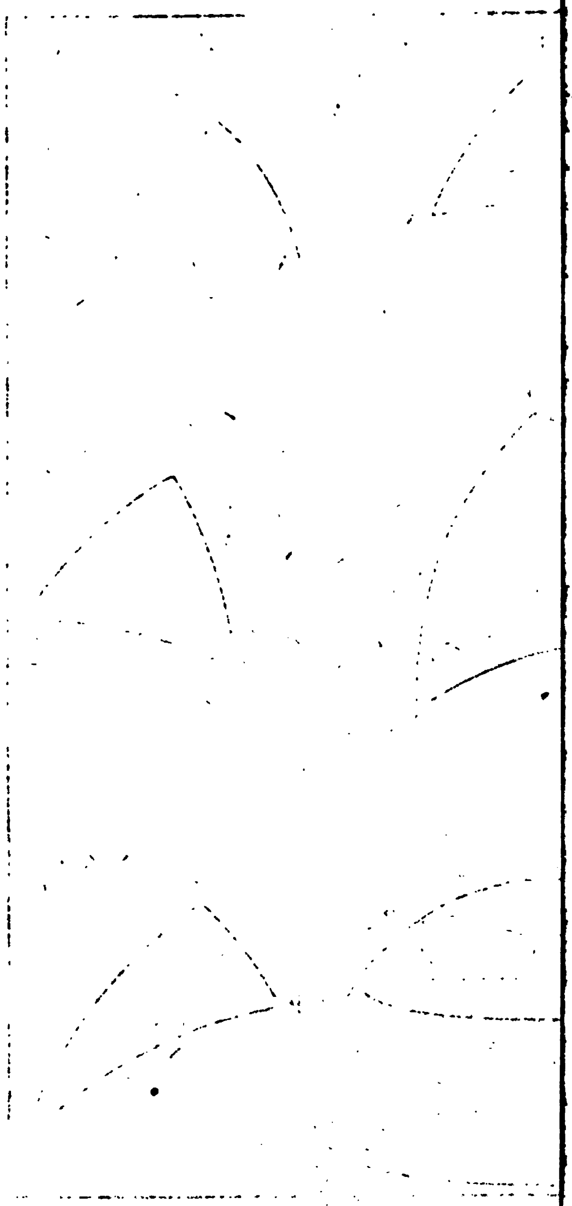
$$\dot{P} = \frac{\dot{\overline{PS}}}{R} \times \overline{\frac{\text{tang. lat.}}{\text{ſin. P}} \pm \frac{\text{tang. dec.}}{\text{tang. P}}}.$$ The half of this expreſſion, reduced into time, is the required correction; the affirmative ſign taking place when the declination is ſoutherly, and the negative ſign when it is northerly. *Q. E. D.*

Remark. For the former of theſe examples (and indeed in a great meaſure for the theory of Trigonometrical Fluxions) I am indebted to Mr. Cotes's Treatiſe, *De æſtimatione errorum in mixtâ matheſi*; and the latter may be ſeen in De la Caille's Aſtronomy. A variety of examples of the applications of Fluxional

Fluxional Analogies are also given in *Les Memoires de l'Academie*, 1774; and in M. De la Lande's Astronomy. But it may not be improper, however, to observe by way of caution to the young reader, that solutions obtained by this method cannot be depended on as strictly true, unless the arcs under consideration be in their evanescent state; and therefore the nearer they approach thereto, the nearer will the solution approach to precision.

The following problems (among many others) are investigated by the *Direct Method of Fluxions*. 1. The Maxima and Minima. 2. The finding of Tangents. 3. Determining the points of contrary Flexure in a Curve. 4. Finding the Evolutes of Curves. 5. Investigating the Catacaustic and Diacaustic Curves. &c. &c. And the *Inverse Method* is applied, 1. To the finding of the Lengths of Curve Lines. 2. To the Quadratures, or finding the Areas of Figures. 3. To investigating the Surfaces of Solids. 4. To determining the Solidities of Bodies; in which is included all Mensuration. 5. To finding the Centers of Gravity, Percussion, and Oscillation. 6. To investigating the Law of Centripetal Force in a given Curve. And lastly, to the solving of all Physical Problems whatsoever. Since then fluxions are universally applicable to all kinds of problems*, I would advise the young algebraist

* There may be cases proposed which are too simple for the method of fluxions, such as finding the areas of parallelograms, the superficies and solidities of parallelopipedons, prisms, cylinders, &c. since a figure or solid similar to these is evidently always assumed in the very nature of the fluxion; and therefore nothing can be inferred from thence. And, on the other hand, there may be cases proposed which seem to be inaccessible by a fluxionary calculus. Such is the problem respecting the equality of the areas of the hyperbola and triangle of equal bases, when the altitudes of both are supposed to be infinite. Here the method of increments answers such



algebraist to endeavour by all means to make himself thoroughly acquainted with them. I know of no book on the subject so well calculated to answer this end as Mr. Emerson's. His forms for fluents are excellent; and nothing can more facilitate the learner's progress than his endeavouring to investigate them.

The following problem having some time ago engaged my attention, by endeavouring to adapt an analytical solution thereto, instead of the trigonometrical one which has hitherto been made use of, I intended here to present the reader with the result of those speculations, not only because the method of solution is new, but as the problem is applicable to some curious phœnomena in Astronomy, particularly in determining with great accuracy the periodic times of the disappearance of Saturn's ring to an observer situated either on the earth or any of the planets; but as there is just published at Paris a treatise, entitled, *Essai sur les Phénomènes relatifs aux disparitions périodiques de l'Anneau de Saturne*, where this problem is solved in a more

such problems with great facility; when fluxions seem to be of no use. Unless, indeed, we may be allowed to reason after the manner of a well-known mathematical writer, in a case almost similar, as follows: Since, by Hypothesis, the last ratio of the areas, or fluents, is that of equality, their fluxions may be also esteemed in that ratio; therefore, if by equating the fluxions, the value of the abciss, or altitude, comes out *infinite*, we have then derived the same conclusion. Thus, in the case proposed, let $t =$ the semitransverse, $c =$ the semiconjugate, and $x =$ the altitude of the triangle; then by the nature of the curve $\frac{c\dot{x}}{t}\sqrt{x^2 - t^2}$ is the fluxion of the area of the hyperbola; and $\frac{cx^2\dot{x}}{2t\sqrt{x^2-t^2}} + \sqrt{x^2-t^2} \times \frac{c\dot{x}}{2t}$ is the fluxion of the area of the triangle. These therefore being equated and reduced, become $x^2 = x^2 - t^2$; hence t is $= o$; but since t is a constant quantity, it can therefore be only *comparatively* nothing to x. Consequently x is infinite; for when a given finite quantity is o when compared with another quantity, the value of that other quantity must be infinite.

general manner, I have thought proper to reserve my own solution for a future publication, and have here given an extract from the second and third sections of the above-mentioned work.

A Problem.

1. Suppose a body T (fig. 20.) revolves in a circle with an uniform motion; and in the same time another body R is uniformly carried in the right line $R\acute{t}S\acute{t}$, passing through S the center of the circle; and that there is given the ratio of the velocity of the body T in the circle to that of the body R in the right line, and also the situation of the latter body in the right line at the instant the former is at $\acute{t}$. Required an expression for the arch described by the body T from its commencement of motion at $\acute{t}$, to the instant they are found in the same perpendicular to the right line $R\acute{t}S\acute{t}$; also an expression for the arch described by T when the body R arrives at S.

Solution.

2. Put $a = \acute{t}R$, $\frac{m}{n} =$ the ratio of the velocity of the body T to that of R, $r =$ the radius of the circle, $u = RQ$, and $z =$ the arch $\acute{t}T\acute{t}P$. By the quest. we have $n : m :: u : z$, $\therefore \frac{mu}{n} = z$; and since PQ is perpendicular to the right line $\acute{t}S\acute{t}$, PQ is the sine, and SQ the cosine of the arch z; hence $Q\acute{t}$ is equal to $r - \text{cos.}z$. And by the construction $Q\acute{t}$ is equal $RQ - \acute{t}R = u - a$, therefore $u - a = r - \text{cos.}z$; which, because

because $u = \frac{nz}{m}$, becomes $\frac{n}{m}z + \text{cos}.z - r - a = 0$. This equation contains every case when the bodies T and R are in the same perpendicular to the line tSt'. When the body R is at S, u is equal $r + a$; therefore in this case the equation becomes $\frac{nz}{m} - r - a = 0$.

3. In the above equations it must be remembered that the sign of cos.z is affirmative when the arch is between 0° and 90°, 270° and 360°; and negative when it is between 90° and 270°. That the arch z may be greater than 360°, and that the sign of the cosine is the same for an arch of $360° + z$ as for the arch z.

4. Before we proceed to investigate the unknown quantity z in these equations, we shall observe, that as the solution will be best exhibited by particular cases, we shall therefore fix on such constant quantities in the equations, as would arise from that part of astronomy to which the question is particularly applicable, *viz.* the Phænomena of Saturn's Ring.

As the distances of the Earth and Saturn from the Sun are respectively as 1 : 9·5394, the length of the Earth's semidiameter will be the sine of 6° 1′ 2″ in Saturn's orbit. Saturn then describeth an arch of 6° 1′ 2″ from the instant the plane of his ring produced touches the Earth's orbit, to the instant it passes through the sun; and as Saturn's periodic time in his orbit is 10761·615 days, he will be 179·83 days in running 6° 1′ 2″. The plane of his ring then taketh nearly 360 days to run the diameter of the earth's orbit; but as this is only the $\frac{113}{355}$th part of it, it is therefore $\frac{355}{113} \times 360$ days in running

the whole orbit; but the earth takes 365·25 days to run the fame distance, hence the velocity of the earth in its orbit is to the simultaneous velocity of the ring of Saturn (considered as in the diameter of the earth's orbit) as $\frac{355}{113} \times 360$ is to 365·25, $\therefore m : n :: \frac{355}{113} \times 360 : 365{\cdot}25 :: 3{\cdot}098 : 1$.

Hence the foregoing equations (the former of which, for the fake of diftinction, we fhall hereafter call the *original equation*) in relation to Saturn will become

$$\frac{1}{3{\cdot}098}z + \text{cof.}\,z - r - a = 0;$$

$$\frac{1}{3{\cdot}098}z - r - a = 0.$$

The firft of thefe equations may have different values; the fecond can only have one value.

5. Now to determine the number of real roots which equations of a fimilar form with the original one may contain in general, we may eafily perceive that they muft depend on the value of a, and the ratio of m to n; and moreover that a cannot be greater than $\frac{n}{m}$ arch 360°, or $\frac{n}{m} \times \frac{2 \times 355}{113} \times r$, this being the fpace the body R defcribes in the right line during one revolution of T. For if a exceeded this quantity, it is evident, that it would become lefs after one revolution of T, which would therefore no ways affect the queftion. It is alfo evident that the equation hath at leaft one real value, fince it is impoffible for the body R to move in the right line Rt, while the body T revolves in the circumference of the circle tTtP, without both being at fome one inftant in the fame perpendicular. But to determine the number of the real values of z, we muft inveftigate fuch values of a whereby the original

original equation may contain two values of z, one real, the other imaginary, for it is by this double value that the roots of the equation in passing from imaginary to real, or from real to imaginary, may be determined.*

6. By

* The demonstration to this proposition I shall give in M. Du Sejour's own words, " *Pour démontrer cette proposition, je me sers du théorême connu de tous les Géometrés, par lequel on enseigne à trouver la valeur analytique d'une fonction, lorsque la variable que cette fonction renferme, croît ou décroît d'une petite quantité.——Soit* φx *une fonction de la variable* x; $dx\Delta x$ *la différence de cette fonction*; $dx\Gamma x$ *la différence de* Δx; $dx\Psi x$ *la différence de* Γx, &c. *Soit ensuite* $\varphi\overline{x+z}$ *une fonction de* $\overline{x+z}$ *pareille à la fonction* φx, z *étant une très petite quantité. On démontre que l'on a*

$$\varphi\overline{x+z} = \varphi x + \frac{z\Delta x}{1} + \frac{z^2\Gamma x}{1\times 2} + \frac{z^3\Psi x}{1\times 2\times 3} + \&c.$$

Appliquons ce principe au cas dont il s'agit. Nous avons en général $a = \frac{n}{m}z + \text{cos}.z - r$. *Réduisons en série chacun des membres de cette équation au moyen du théorême précédent. Il est aisé de voir que si l'on suppose que, tandis que la variable* a *qui entre dans le premier membre de l'équation, croît d'une petite quantité* α, *la variable* z *qui entre dans le second membre, croît d'une autre petite quantité* β, *l'on aura en vertu du théorême,*

$$a + \alpha = \frac{n}{m}z + \text{cos}.z - r + \beta \times \frac{\frac{n}{m}r - \text{sin}.z}{r} - \frac{\beta^2\text{cos}.z}{1.2r^2} + \frac{\beta^3\text{sin}.z}{1.2.3r^3}, \&c.$$

Mais par la supposition, $a = \frac{n}{m}z + \text{cos}.z - r$; *de plus puisqu'il s'agit des points particuliers qui répondent aux racines doubles de l'equation, l'on a* (§ 6.) $\frac{n}{m}r - \text{sin}.z = 0$; *donc dans le cas où la quantité* β *est infiniment petite, l'on a*

$$2\alpha r^2 + \beta^2 \text{cos}.z = 0,$$

ou ce qui revient au même,

$$\beta = r\sqrt{\frac{-2\alpha}{\text{cos}.z}}.$$

Donc β *est demi-imaginaire & dépend du signe de* α. *Donc c'est par la valeur particulière de* a *correspondante aux racines doubles de l'équation, que cette équation acquiert de nouvelles valeurs réeles, & réciproquement; puisqu'en*

de-ça

6. By the method *De Maximis et Minimis* we find that the equation hath a double value when $\frac{n}{m}dz + d \times \text{cof.}\,z = 0$, or $\left(\text{becaufe } d \times \text{cof.}\,z = -\frac{\text{fin.}\,z.d.z}{r}\right) \frac{nr}{m} - \text{fin.}\,z = 0$. Hence the equation will have double roots when the bodies T and R are in the fame perpendicular, and the former hath defcribed an arch the fine of which is expreffed by $\frac{nr}{m}$, which may be greater or lefs than 90°; or that arch augmented by 360 degrees, or any multiple thereof.

7. As the fine of an arch can never exceed the radius, it is evident becaufe of the condition $\frac{nr}{m} - \text{fin.}\,z = 0$, that the equation can only have double roots when n is lefs than m, or at moft its equal. Let us fuppofe $n = m$, that is, let the bodies R and T have the fame velocity, then we fhall have $\text{fin.}\,z = r$, and the correfponding values of z and $\text{cof.}\,z$, will be *arch* 90°, and 0. Therefore in the fuppofition of $n = m$, Equations of a fimilar form with the original one can only have double values when $a = \text{arch } 90° - r$.

de-ça de cette valeur β eft réel, & qu'au de-là de cette valeur β eft imaginaire, ou réciproquement.

On démontre par la même analyfe, que l'équation ne peut acquérir de nouvelles valeurs réelles, que par les points particuliers qui répondent aux racines doubles; en effet s'il n'étoit pas queftion de ces points, la quantité $\frac{nr}{m} - \text{fin.}\,z$ *ne feroit pas nulle; & l'on auroit en vertu du théoréme, dans la cas où β eft une quantité infiniment petite,* $\alpha = \beta \times \frac{\frac{nr}{m} - \text{fin.}\,z}{r}$. *D'où l'on voit que foit que l'on fuppofe α pofitif ou négatif, l'on aura toujours une valeur de β réelle.*

8. In

8. In order to exhibit the roots of the equation in the supposition of $n = m$, we shall represent the different values of a by the abcisses of a curve, and the corresponding roots $\frac{nz}{m}$ by the ordinates of the same curve. The locus of the original equation will be a curve m, m', m'' (fig. 21.) having a point of inflexion at m', the ordinate to this point will touch the curve, and join three points thereof; hence the equation will have three equal roots, and the corresponding abcis AP$'$, or a, will be expressed by, arch $90° - r$. If we take any other point m in the curve, the corresponding ordinate AP will be expressed by a simple equation, and therefore have only one real root. And in the supposition of n being greater than m, the curve will have no point of inflexion; for whatever the value of a is, the ordinate will be expressed by a simple equation, containing but one real root. If the curve be considered as purely geometrical, not only the part m, m', m'', but also the part m'', m''', m'''', equally resolveth the question; but though the inflexions are similar, yet as the abcisses AP$'''$, AP$''''$, &c. are expressed by $360° + 90° - r$, $2 \times 360° + 90° - r$, &c. they are of no use in the astronomical part of the solution, as they exceed 360°, agreeable to what we observed in § 5. wherefore the part of the curve m, m', m'', can only satisfy the problem.

9. If n be less than m the locus of the equation will be a curve of various sinuosities (as in fig. 22, 23.) depending on the different relations of m to n. And *one* or *more* of these sinuosities (formed by the different returns of the curve) may be cut by the same ordinate line, according as the ratio of m to n approaches to, or recedes from, the ratio of equality.

10. To illustrate this, let us suppose the ratio of m to n as 10 to 9, and we shall have by the equation $\frac{nr}{m} - \text{sin.}\, z = 0$, $z = \frac{9r}{10}$; from whence it appears that the angles $64^\circ, 9', 30''$, $115^\circ, 50', 30''$, answer the conditions of the question; the cosine of the former arch being $+ \frac{r\sqrt{19}}{10} = + 43589$, and of the latter $- 43589$. These values of the arch z and its cosine being substituted in the original equation, give

$$\frac{9}{10} \text{ arch } 115^\circ\ 50'\ 30'' - 43589 - 100000 - a = 0,$$

$$\frac{9}{10} \text{ arch } 64^\circ\ 9'\ 30'' + 43589 - 100000 - a = 0.$$

From the former of which equations we get $a = 38376$; and from the latter $a = 44369$. Other values of a may also be found from the equations

$$\frac{9}{10} \times \overline{360^\circ + 115^\circ, 50', 30''} - 43589 - 100000 - a = 0,$$

$$\frac{9}{10} \times \overline{360^\circ + 64^\circ, 9', 30''} + 43589 - 100000 - a = 0,$$

from whence $a = 603863$, and 609856.

11. Since when $m : n :: 10 : 9$, the equation $\frac{n}{m} z + \text{cos.}\, z - r - a = 0$, can only have three real roots when the value of a is between 38376 and 44369, it appears that if the ratio of the velocities of the two bodies T and R be as 10 to 9, and that if at the instant the body T is at t' the body R is also in the same point, they will meet but once in the same perpendicular; but if at that time R is at some distance from t' between 38376 and 44369, they will be three times in the same perpendicular; and if the distance be greater than the last number, they can meet but once.

12. If

12. If m have a much greater ratio to n than in the preceding supposition, the two angles z by which the original equation acquireth double roots, will be much farther removed from 90°, the one approaching nearer to 180°, and the other to 0°. If we suppose the angle z determined by the equation $z = \frac{nr}{m}$ to be always the least of the angles resulting from this relation, the original equation will in this case be of equal value with the following ones,

$$\frac{n}{m} \times \overline{180^\circ - z} - \text{cos.}\, z - r - a = 0,$$

$$\frac{n}{m} z + \text{cos.}\, z - r - a = 0,$$

The first of these equations shews the relation between the abcifs and ordinate at the extremity m of the sinuosity A$\acute{m}m$ (fig. 24.), and the second expresseth *that* between the abcifs and ordinate at the other extremity of the same sinuosity $\acute{m}$. If then we suppose that the point of the curve m answereth to the abcifs $= 0$, the first equation will become

$$\frac{n}{m} \times \overline{180^\circ - z} - \text{cos.}\, z - r = 0,$$

which by substituting for $\frac{n}{m}$ its equal $\frac{\text{sin.}\, z}{r}$ becomes

$$\overline{108^\circ - z} \times \overline{\text{sin.}\, z - r} \times \overline{r + \text{cos.}\, z} = 0;$$

therefore the problem is determined, depending on the quadrature of the circle. From these equations we easily find

$$z = \text{arch } 46^\circ, 26', 0''; \; 180^\circ - z = 133^\circ, 34', 0'';$$

$$\frac{n}{m} = \frac{7246}{10000} = \frac{1}{1{\cdot}38}.$$

13. Hence it appears that if the ratio of the motions of the two bodies be as 10000 is to 7246, and that at the commencement of motion they are both at the point $\acute{t}$, they will again meet

meet when T has described an arch of $133^{\circ}, 34', 0''$; which second meeting will evidently be double. If the value of a is between 0 and 27640, there will be three meetings of the two bodies in the same perpendicular; but if a be greater than this number they can meet only once.

14. Not only when the value of a is between 0 and 27640, but when it hath the particular value 455262 (this being the last value of a which should be considered, § 5.), the bodies R and T will meet three times in the same perpendicular; which will be evident if we consider that this is the distance the body R has run during one revolution of T in its orbit. Let A (fig. 25.) be the origin of the curve $A\,m'\,m''\,m'''\,m$, to which the abcifs is $= 0$; the abcisses AP, AP'', respectively equal 27640, 455262. Now it is evident that the sinuosities of the curve are bounded by the ordinates Am, $P'm'$, $P''m''$; and as the sinuosity $Am'm$ is limited by the ordinate Am corresponding to the abcifs $= 0$, so is the sinuosity $m''m'''m''$ limited by the ordinate $P''m''$ corresponding to the abcifs AP''. In this case, while the part of the curve $m''m'''m''$ beginneth to appertain to the question, the analogous part $Am'm$ ceaseth to resolve it.

15. If the ratio of m to n be still farther distant from that of equality, the sinuosity $Am'm$ (fig. 26,) will not be limited by the ordinate Am corresponding to the abcifs $= 0$, as in the preceding case, but extendeth itself over the negative side of the abcifs; which neverthelefs is one part of the curve which answers the conditions of the question. In like manner the

the sinuosity $m'''m''m'''$ extends itself beyond the ordinate $P'''m'''$ answering to the last abcifs AP''', and is another part of the curve which satisfieth the question. If in this case we denote the abcifs at the origin of the curve by o; the last abcifs, (being that which results from § 5.) by a'''; the two values of a determined by the equations $\frac{n}{m}z + \text{cos.}\,z - r - a = 0$, $\frac{n}{m} \times \overline{360° + 180° - z} - \text{cos.}\,z - r - a \mp 0$, by a', a'', respectively; and the least angle determined by the equation $z \mp \frac{nr}{m}$ by z; then will the two bodies meet three times in the same perpendicular when the value of a is between o and a', and between a'' and a'''; and only once when it is between a' and a''*.

16. We

* This is a remarkable case, and should be particularly attended to, because it is only from the supposition of $a = 0$ that we can determine precisely the number of real roots of the equation. For when we have discovered by the ratio of m to n that the equation is capable of containing a certain number of real roots, we only know in general that it hath essentially this number of real roots, except two; these two depending on the value of a. It is necessary then to have some distinguishing character, whereby we can ascertain the reality of the two roots depending thereon. Now this characteristic is not constant, but changes in that particular case when the supposition of $a = 0$ giveth double roots to the equation. For instance, when the value of $\frac{m}{n}$ is between $\frac{1}{1}$ and $\frac{10000}{2173}$, the original equation may have either *three* real roots or only *one*, this circumstance depending on the value of a. The particular value of $\frac{m}{n} = \frac{10000}{7246}$ giveth the equation two equal roots (§ 12.) in the case of $a = 0$; and it is in this point that the characteristic of a changes. When the value of $\frac{m}{n}$ is between $\frac{1}{1}$ and $\frac{10000}{7246}$, if we denote the two values of a determined by the

equations

16. We shall now shew what must be the ratio of m to n, that the equation $\frac{n}{m} z + \text{cos.} z - r - a = 0$ may have five real roots.

When the sinuosities of the curve are so far extended that the same ordinate touches the extremities of two of them as in fig. 27. the curve hath then essentially three real roots; and at the particular point corresponding to the abcifs $A\acute{P}$ it hath five roots. At this point the curve passeth from *one* essential real root and two other possible ones to *three* essential real roots and two other possible ones; and to determine the particular condition which then taketh place, we must observe, that the equation $\frac{n}{m} z + \text{cos.} z - r - a = 0$ expresseth the

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equations $\frac{n}{m} \times 180° - z - \text{cos.} z - r - a = 0$, $\frac{n}{m} z + \text{cos.} z - r - a = 0$, by $\acute{a}$, $\acute{\acute{a}}$, respectively, and by o, $\acute{\acute{a}}$, and z, the same as in § 15, the bodies will meet *three* times when a is between $\acute{a}$ and $\acute{\acute{a}}$, and only *once* when a is between o and $\acute{a}$, $\acute{\acute{a}}$ and $\acute{\acute{\acute{a}}}$. But on the contrary when the value of $\frac{m}{n}$ is between $\frac{10000}{7246}$ and $\frac{10000}{2173}$, the characteristics of a are different; for in this case $\acute{a}$ must be determined by the equation $\frac{n}{m} z + \text{cos.} z - r - a = 0$; and $\acute{\acute{a}}$ by the equation $\frac{n}{m} \times 360° + 180° - z - \text{cos.} z - r - a = 0$; when the bodies will meet *three* times when a is between o and $\acute{a}$, $\acute{\acute{a}}$ and $\acute{\acute{\acute{a}}}$; and only *once* when a is between $\acute{a}$ and $\acute{\acute{a}}$.——It may perhaps appear to the reader, that when the characteristic of a changes, the particular values thereof might be determined from changing the order of these equations, that is by a reciprocal substitution of one case for the other; but from hence we should find either that some values of a were negative, or some greater than those which result from § 5. which therefore ought to be rejected. Hence we see the inconvenience attending such a procedure.

relation

relation between the abcifs and ordinate at the extremity m of the finuofity $A\,m'\,m$; and the equation

$\frac{n}{m} \times \overline{360^\circ + 180^\circ - z} - \text{cof.}\, z - r - a = 0$ determines the relation between the abcifs and ordinate at the extremity of the finuofity $m''' m'' m'''$. Now becaufe (by the hypothefis) the two finuofities have the fame tangent $P\,m'\,m''$, or rather becaufe they coincide at that point, we have at the fame time the two following equations,

$$\frac{n}{m} z + \text{cof.}\, z - r - a = 0;$$

$$\frac{n}{m} \times \overline{360^\circ + 180^\circ - z} - \text{cof.}\, z - r - a = 0;$$

which, by equating the two values of a derived from hence, give for the condition of the problem

$$\frac{n}{m} \times \overline{\frac{360^\circ + 180^\circ}{2} - z} - \text{cof.}\, z = 0;$$

or, becaufe $\frac{n}{m} = \frac{\text{fin.}\, z}{r}$,

$$\text{fin.}\, z \times \overline{\frac{300 + 180^\circ}{2} - z} - r\, \text{cof.}\, z = 0.$$

An equation depending on the quadrature of the circle. From hence we eafily find

$z =$ arch $12^\circ, 32', 50''$, $\frac{n}{m} = \frac{2173}{10000} = \frac{1}{4{\cdot}603}$, and $a = {\cdot}2371$.

17. Hence it appears that if the ratio of the motions of the two bodies T and R be as 10000 is to 2173, and that when the body T is at t', R is diftant ·2371, they will meet five times in the fame perpendicular. In all other circumftances they can meet only three times.

18. We

18. We shall now enquire what the ratio of m to n ought to be, when a is supposed $= 0$, that the equation $\frac{n}{m} z + \text{cos.}\, z - r - a = 0$ may have five real roots.

If the ratio of m to n be yet farther distant from that of equality, the sinuosity $\overset{'''}{m}\overset{''}{m}\overset{'''}{m}$ is not limited by the ordinate $P\overset{'}{m}\overset{''}{m}$, but passeth beyond it; and there is one particular case where it touches the ordinate corresponding to the abcis $= 0$. To determine which we must observe that the equation

$$\frac{n}{m} \times \overline{360° + 180° - z} - \text{cos.}\, z - r - a = 0,$$

expresseth the relation between the abcis and ordinate at the extremity $\overset{''}{m}$ of the sinuosity $\overset{'''}{m}\overset{''}{m}\overset{'''}{m}$ (agreeable to § 16.) If then we suppose that the point $\overset{''}{m}$ in the curve hath its abcis $= 0$, we shall have $a = 0$ in the last equation, therefore

$$\frac{n}{m} \times \overline{360° + 180° - z} - \text{cos.}\, z - r = 0;$$

but $\frac{n}{m} = \frac{\text{sin.}\, z}{r}$, hence by substitution,

$$\overline{360° + 180° - z} \times \text{sin.}\, z - r \times \overline{r + \text{cos.}\, z} = 0,$$

an equation depending still on the quadrature of the circle. From hence we get

$$z = 12°, 23', 48''; \; 360° + 180° - z = 527°, 36', 12'';$$

$$\frac{n}{m} = \frac{2147}{10000} = \frac{1}{4{\cdot}658}.$$

19. From whence it appears that if the ratio of the motions of T and R be as 10000 to 2147, and that they both begin to move at the same time from the point $\overset{'}{t}$, they will first be

in

in the same perpendicular at the commencement of motion, a second and third time when T hath described two certain arches, and a fourth time when it hath described an arch of $527^{\circ}, 36', 12''$, from the point t', and this last meeting will be double. If the value of a be contained between 0 and 2341, determined by the equation $\frac{n}{m} z + \text{cos}. z - r - a = 0$, there will be five meetings of the two bodies in the same perpendicular; but if a be greater than this last number they can meet only three times. But beside the above limit for a, the particular value 134920 giveth yet five meetings of the bodies in the same perpendicular; this being the distance run by R during one revolution of T in its orbit, agreeable to what has been already observed in § 5. Let A (fig. 28.) be the origin of the curve, to which the correspondent abcis is $= 0$, and the abcisses AP', AP'', respectively equal to 2314 and 134920. Here then it is evident that the sinuosities of the curve are bounded by the ordinates Am, $P'm'$, $P''m''$; and in the same manner as the sinuosity $m'' m\, m''$ is limited by the ordinate Am of which the abcis is 0, so is the sinuosity $m''' m'' m'''$ limited by the ordinate $P''m''$ of which the abscis A is 134920. This is then the case where the part of the curve $m''' m'' m'''$ beginneth to answer the question considered astronomically, while the analogous part of the curve $m'' m\, m''$ ceaseth to resolve it.

20. If the ratio of m to n continueth to be still farther removed from the ratio of equality, the sinuosity of the curve $m'' m\, m''$ will then not be limited by the ordinate Am corresponding to the abcis $= 0$, but will extend itself over the negative side of the abcis, as we have before observed in § 15,

and

and in a similar manner will the sinuosity $\overset{\prime\prime\prime}{m}\,\overset{\prime\prime}{m}\,\overset{\prime}{m}$ extend itself beyond the ordinate $P\overset{\prime\prime}{m}$ corresponding to the last abcifs $A\overset{\prime\prime}{P}$. If in this case we denote the first abcifs by o; the last abcifs derived from § 5, by $\overset{\prime\prime\prime}{a}$; the two values of a respectively determined by the equations $\frac{n}{m} z + \text{cos.}\, z - r - a = 0$; $\frac{n}{m} \times \overline{2 \times 360^\circ + 180^\circ - z} - \text{cos.}\, z - r - a = 0$, by $\overset{\prime}{a}, \overset{\prime\prime}{a}$; and the least angle deduced from the equation $z = \frac{nr}{m}$ by z; the two bodies will meet five times when a is between o and $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$ and $\overset{\prime\prime\prime}{a}$; but will meet only three times when it is between $\overset{\prime}{a}$ and $\overset{\prime\prime}{a}$. If the value a be comprized between $\frac{10000}{2173}$ and $\frac{10000}{2147}$, and if $\overset{\prime}{a}, \overset{\prime\prime}{a}$, respectively represent the two values of a determined by the equations

$$\frac{n}{m} \times \overline{360^\circ + 180^\circ - z} - \text{cos.}\, z - r - a = 0,$$

$\frac{n}{m} z + \text{cos.}\, z - r - a = 0$, the bodies R and T will meet but three times when a is between o and $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$ and $\overset{\prime\prime\prime}{a}$; and five times when between $\overset{\prime}{a}$ and $\overset{\prime\prime}{a}$.

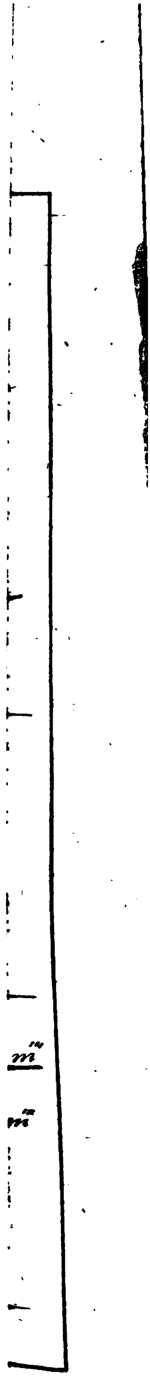
21. If we would investigate the ratio of m to n, so that the original equation might have seven real roots, we should find for the condition of the problem

$$\text{sin.}\, z \times \overline{\frac{2 \times 360 + 180^\circ}{2} - z} - r\, \text{cos.}\, z = 0;$$

from whence we may easily find $z = \text{arch } 7^\circ, 22', 31''$;

$$\frac{n}{m} = \frac{1284}{10000} = \frac{1}{7{\cdot}791};\ a = 788.$$

And


m''
m''

And in the ſame manner we might proceed to determine the ratio of m to n, ſo that the original equation would be ſuſceptible of any aſſigned number of real roots.

22. We ſhall now exhibit in a general manner, how to determine in all caſes the number of real roots of the original equation.

Write down in the following order the equations which have been deduced in the preceding Articles;

$$1.\ \frac{\text{ſin.}\,z}{r}z + \text{coſ.}\,z - r - a = 0;$$

$$2.\ \frac{\text{ſin.}\,z}{r} \times \overline{180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

$$3.\ \frac{\text{ſin.}\,z}{r} \times \overline{360^\circ + 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

$$4.\ \frac{\text{ſin.}\,z}{r} \times \overline{2 \times 360^\circ + 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

$$5.\ \frac{\text{ſin.}\,z}{r} \times \overline{3 \times 360^\circ + 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

$$6.\ \frac{\text{ſin.}\,z}{r} \times \overline{4 \times 360^\circ + 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

&c.

Theſe ſhew the relation between the abciſs and ordinate correſponding to the extreme points of the different ſinuoſities of the curve. And if we ſuppoſe z to repreſent the leaſt angle determined by the equation $\frac{\text{ſin.}\,z}{r} = \frac{n}{m}$, we have ſeen that the compariſon of the firſt and ſecond equations ſheweth the circumſtance when the caſe propoſed hath *one* real root and beginneth to have *three*; and the equation to effect this is,

$$\text{ſin.}\,z \times \overline{\frac{180^\circ}{2} - z} - r\,\text{coſ.}\,z = 0.$$

 The

The comparison of the first and third equations giveth the circumstance when the case proposed hath essentially *three* real roots and beginneth to have *five*; the condition of which is,

$$\text{sin.}\, z \times \overline{\frac{360^\circ + 180^\circ}{2} - z} - r\ \text{cos.}\, z = 0.$$

The comparison of the first and fourth equations determineth likewise the circumstance when the case proposed hath essentially *five* real roots, and beginneth to have *seven*; and the condition of this is

$$\text{sin.}\, z \times \overline{\frac{2 \times 360^\circ + 180^\circ}{2} - z} - r\ \text{cos.}\, z = 0.$$

And as the analogy is established, it is evident that if we put $\nu =$ the number of real roots the equation can have, we may easily determine the circumstance when the equation proposed beginneth to have this number of real roots, by the general expression,

$$\text{sin.}\, z \times \overline{\frac{\nu - 2}{2} \times 180^\circ - z} - r\ \text{cos.}\, z = 0.$$

We may likewise determine the circumstance when the equation hath *essentially* ν real roots, by the expression

$$\text{sin.}\, z \times \overline{\frac{\nu}{2} \times 180^\circ - z} - r\ \text{cos.}\, z = 0.$$

In all these cases the corresponding value of $\frac{n}{m}$ is given from the equations $\frac{n}{m} = \frac{\text{sin.}\, z}{r}$.

23. From hence it would be no difficult matter to form a table, which should shew by inspection from the value of $\frac{n}{m}$, what number of real roots the equation contains. Now in order to ascertain this matter truly, we must have regard to the value

value of a, ſince two of the roots depend thereon. But it has been already ſhewn that when one value of a is denoted by o, the value of a determined by § 5. by a''', and any two intermediate values of a by a', a''; the combinations of the values o, a', a'', a''', will clearly exhibit the nature of *all* the roots of the equation. In the caſe, for example, where the equation hath eſſentially *one* real root, and may have *three* real ones, the values of a', a'', have been given by two of the three equations of § 22, that is by the equations (1) and (2) in one circumſtance, and by the equations (1) and (3) in another circumſtance; for the equation (1) always holds good, but the equations (2) and (3) cannot both hold good at the ſame time. We have ſeen alſo when the equation hath eſſentially *three* real roots, and may have *five* real ones, that the values of a', a'', have been given from two of the three equations (1), (3), (4), § 22, that is by the equations (3) and (1) in one circumſtance, and by the equations (1) and (4) in another circumſtance; the equation (1) always obtaining, while the equations (3) and (4) cannot both obtain at the ſame. As the analogy is then evident, we may conclude, that if r repreſents the number of real roots the propoſed equation may contain, as before, the following equation will determine the values of a', a'',

$$1.\ \frac{\text{ſin.}\,z}{r}+\text{coſ.}\,z-r-a=0,$$

and the only uncertainty remaining is, which of the two following equations we muſt make uſe of,

$$2.\ \frac{\text{ſin.}\,z}{r}\times\overline{r-2\times180^\circ-z}-\text{coſ.}\,z-r-a=0;$$

$$3.\ \frac{\text{ſin.}\,z}{r}\times\overline{r\times180^\circ-z}-\text{coſ.}\,z-r-a=0;$$

in

in which equations we must remember that z is determined by the equation $\frac{\text{sin.}\,z}{r} = \frac{n}{m}$, and that $\frac{n}{m}$ is given in the question.

24. This last difficulty may be easily obviated; for since the equations (2) and (3) § 23. cannot both obtain at the same time, if the equation (2) for instance, be that which we ought to use, the equation (3) will give essentially a value of a greater than a'''; but if the equation (3) be that we should make use of, the equation (2) will give a value of a less than o. We should then immediately try the equation (2), and if it give a value of a greater than o, this is the value we have denoted by a', and the equation (1) will give a''. If on the contrary the equation (2) giveth a value of a less than o, it is of no use, and the equation (1) will determine a', as the equation (3) will give a''.

In the first case the proposed equation will have $\nu - 2$ real roots when the value of a is between o and a', a'' and a'''; and ν real roots when a is between a' and a''. In the second case, the equation will have ν real roots, when a is between o and a', a'' and a'''; and $\nu - 2$ real roots, when a is between a' and a''.

26. The table we have before mentioned for determining the number of real roots in the equation from the values of $\frac{n}{m}$, may be conveniently drawn up in the following form,

$\frac{n}{m} = 1$

$\frac{n}{m} \sqsubset 1$ to $\frac{n}{m} = 1$	The equation hath essentially one real root, and can have no more, whatever the value of a is.
$\frac{n}{m} = 1$ to $\frac{n}{m} = \frac{1}{4{\cdot}603}$	One essential real root, and may have three, according to the different values of a.
$\frac{n}{m} = \frac{1}{4{\cdot}603}$ to $\frac{n}{m} = \frac{1}{7{\cdot}791}$	Three essential real roots, and may have five according to the different values of a.
$\frac{n}{m} = \frac{1}{7{\cdot}791}$ to $\frac{n}{m} = \frac{1}{10{\cdot}951}$	Five essential real roots, and may have seven, by the different values of a.

And in the same manner may the table be continued as far as we please, by means of the equations of § 22.

The application of the solution of this problem to determining the times of the appearance and disappearance of saturn's ring, will be obvious to any one who has attentively considered the subject; those who have not, will not think their time ill spent in perusing M. Du Sejour's curious treatise, from whence I have before observed this solution is chiefly extracted.——The requisites for the application, as deduced from M. De le Lande's Astronomy, are as follow,

Mean distance of the earth from the sun - - -	100000.
Mean distance of saturn from the sun - - - -	953937.
Syderial revolution of the earth - - - -	365^{d}, 6^{h}, $9'$.
Syderial revolution of saturn - - - -	10761, 14, 37.

 Time

Time of the plane of the ring passing through the sun, -	1774, *Jan.* 8th,
The earth's place in its orbit at that time, -	3s, 18°, 23′, 0″,
Longitude of the ascending node of saturn's orbit	3, 21, 43, 17.
Longitude of the node of the ring - - -	5, 17, 5, 0.
Inclination of the ring to the ecliptic - - -	31°, 20, 0,
Inclination of saturn's orbit to the ecliptic - -	2, 30, 20.
Longitude of those points of the earth's orbit which the plane of the ring touches when produced,	2, 17, 5, 0, 8, 17, 5, 0.

Some new Geometrical Propositions, which will be often found useful in the solutions of Problems.

Proposition I. Theorem. Fig. 29.

If from the middle of the chord of a given segment of a circle ADB, a perpendicular be drawn to D, and AD produced to E, so that DE be equal to AD, and through E another segment be described on AB; then, if a line be drawn from the end of the chord as AF, the part intercepted between the two peripheries GF will be always equal to GB. ——For DB is evidently = DE; and AFB is = AEB, also AGB = ADB (Euc. 21. 3.) therefore EDB = FGB, and the triangles BDE, BGF are similar, consequently BG = GF.

Proposition II. Problem.

The same construction being made as in the last proposition, it is required so to draw the line AF, that either the sum, difference, ratio, rectangle, sum of the squares, or difference of the squares of AG and GF, may be of a given magnitude,

For

For the Sum.——With diſtance AF = the given ſum deſcribe an arch to cut the periphery of the greater ſegment in F, and join A, F; and AG + GF will be = the given ſum. This is evident; as alſo the limitation,——that the given ſum muſt not be greater than 2AD, nor leſs than AB.

The Difference.——With diſtance AL = the given difference of AG, GF, deſcribe an arch *a b*; and on AB deſcribe the ſegment of a circle containing an angle equal to the ſup. of DAB, (Euc. 33. 3.) and through the interſection L draw ALF; ſo will AG − GF = AL the given difference. ——For ſince AL is equal the given difference *per* con. GF will be equal GL; but GF = GB *per* con. therefore GL = GB; hence ALB = LGB + GBL (Euc. 32. 1.) = ADB + DBA = ſup. of DAB.——The limitation is evident, ——the given difference muſt not be greater than the diameter of the leſſer circle.

The Ratio.——Divide AB in the given ratio of AG to GF in O, complete the circle ADB (Euc. 25. 3.), and produce DC to P; then through O draw PG, and through G draw AF; and AG will be to GF in the given ratio.——For ſince the arches AP, BP, are equal, the angle AGB is biſected by GP (Euc. 27. 3.) and therefore as AO : OB :: AG : GB :: AG : GF, *per* Euc. 3. 6. and conſtruction.

The ſame may be effected independent of the property of the circle,—Having divided AB in the given ratio in O, in any angle draw the indefinite line AH, and take AH to HI alſo in the given ratio; join IO, parallel to which draw HK, and from center K and diſtance KO deſcribe the arch OG; then draw AGF, and AF will be divided in the given ratio in G.——For by conſtruction AH : IH :: AO : OB, and becauſe of the parallels IO, HK, AH : HI :: KA : KO.

 ∴ KA

$\therefore$ KA : KO :: AO : OB, and by alternation and division KA : KA − AO :: KO : KO − OB, that is KA : KO :: KO : KB, or KA : KG :: KG : KB; hence the triangles AKG, GKB are similar (Euc. 6. 6.) and the other sides will be proportional, or AG : GB :: AK : KG (KO) :: AH : HI, which are in the given ratio by construction.

The Rectangle.——Let M represent the side of a square equal to the given rectangle (Euc. 14. 2.). Find the centre of the lesser circle V (Euc. 25. 3.) and on AB produced take AK equal to the diameter thereof; make the normal KT a third proportional to AK and M, and draw TG || to AK; then through G draw AGF, and AG × GF will be equal M^2.——For through the center V draw GS, join SB, and demit the ⊥ GR; then are the triangles AGR, SGB similar (Euc. 21. 31. 3.) and AG : GR :: SG : GB, hence AG × GB = GR × SG; but AK = SG *per* con. and TK = GR, $\therefore$ AG × GB = AG × GF = AK × TK = M^2 *per* construction.——Limitation. M^2 must not exceed AD × DE, or the square of the semi-diameter of the greater circle.

The Sum of the Squares.——Let $2M^2$ express the sum of the squares, and take $CQ^2 = M^2 - AC^2$; with distance CQ and center C describe the arch QG, and through G draw AF, and $AG^2 + GF^2$ will equal $2M^2$.——For $AQ^2 = QC^2 + AC^2$ (Euc. 47. 1.) and $AQ^2 + QB^2 = 2AQ^2 = 2QC^2 + 2AC^2$; also $AG^2 + GB^2 =$
$AC^2 + CG^2 + 2AC \times CR + AC^2 + CG^2 - 2AC \times CR$ (Euc. 12. 13. 2.) $= 2AC^2 + 2CG^2$; but CG = CQ *per* con. therefore $AG^2 + GB^2 = AG^2 + GF^2 = 2AQ^2 = 2CQ^2 + 2AC^2 = 2M^2$ *per* construction.——Limitation. M^2 must not be greater than $AC^2 + CD^2$, nor less than $\frac{1}{2}AB^2$.

The

The Difference of the Squares.———Let M^2 = the given difference of the squares, and take CR = to half a third proportional to AB and M; then draw RG $\perp$ to AB and through the intersection G draw AF; so will $AG^2 - GF^2 = M^2$. ——For $AG^2 - GB^2$ is $= \overline{AR^2 + GR^2} - \overline{BR^2 + GR^2}$ (Euc. 47. 1.) $= AR - BR^2 = \overline{AR + RB} \times \overline{AR - RB}$ (Euc. 5. 2.) = (because AR evidently exceeds RB by 2CR) $AB \times 2CR = M^2$ *per* construction.——Limitation. M must not exceed AB.

Proposition III. Theorem. Fig. 30.

If the base of a triangle ABC be produced both ways to D and E, so that CE = CB, and AD = AB; and the center of a circle F be found to pass through the points D, B, E, (Euc. 5. 4.) and B,F be joined, the angle ABC will be bisected by BF.——For join BD, BE, FA, FD and FE, then is ACB = CBE + BEC = 2BEC (by con. and Euc. 32. 1.) = BFD (Euc. 20. 3.), and in like manner BAC = BFE; hence ABC = EDF + DEF = 2FDA; and because the triangles BAF, DAF are evidently equal and similar, ABF = FDA = $\frac{1}{2}$ABC, therefore ABC is bisected by BF.

Proposition IV. Problem. Fig. 31.

There is given DC the distance of the indefinite line AB from the center of the given circle IFE; it is required so to draw a line through I, that the part intercepted between AB and the concave periphery of the circle may be of a given length.

On IE produced (if necessary) take IP equal to the given line, on which describe the semicircle IOP. Make IS a mean proportional between CI and IE, and $\perp$ to CE; draw SO $\parallel$ to CE, and from I apply IL = SO, which produce to F, and

LF

LF will be equal the given line.——For demit the normal OT, and join F, E. Then becaufe ICL, IFE are right angles, and CIL = FIE (Euc. 15. 1.) the triangles ICL, IFE are fimilar; therefore CI : IF :: LI : IE, hence CI × IE = LI × IF; but SI² = CI × IE *per* con. alfo IP = the given line *per* con. ∴ IT × TP = TO² (Euc. 13. 6.) = SI² = LI × IF; and LI = IT (= CO) *per* con. confequently IF = TP, and LF = IP the given line *per* conftruction.

Propofition V. Theorem. Fig. 32.

If two circles ACD, FDG touch each other, and another circle be any where defcribed cutting both the former circles, as in the points A, C, G, F, the chords AC, FG being produced will meet the tangent drawn from the point of contact D in the fame point B.——For in the circle ACGF, AB × BC = FB × BG; and in the circle ACD, AB × BC = BD²; alfo in the circle FDG, FB × BG = BD² (Euc. 36. 3.) hence DB is a common tangent to both circles, and confequently the lines AB, DB, FB meet in the fame point B.

The converfe of this is alfo true.——That if two lines be any how drawn from a point B to cut a circle ACGF in C, A, G, F; and upon one of the chords as GF a circle be defcribed FDG, and from the point B a tangent BD be drawn to this circle; then, if on ED produced the center of a circle be taken to pafs through the points A and C, it will alfo pafs through the point D, or touch the circle FDG in D.——For EDB is a right angle (Euc. 18. 3.) and therefore MDB is a right angle; and becaufe DM paffes through the center of the circle M, and BD is drawn from a point B on a line AB cutting this circle, and moreover that AB × BC = FB × BG = BD², the line BD touches the circle M in D (Euc. 18. 38. 3.) confequently the circle ACD touches the circle FDG in D.

Propo-

Propoſition VI. Problem. Fig. 33.

It is required to find the center of a circle on a line AC given in poſition, which ſhall paſs through a given point on that line, and touch a circle given in magnitude and poſition.

From the given point A to the center of the given circle draw AB, make BF ∥ to AC, and join A, F; then through E draw BC, and C will be the center of the required circle. ——For FEB = CEA, and becauſe of parallels EBF = ACE, or EFB = CAE, hence the triangles EBF, ACE are ſimilar; but BE = BF, therefore CA = CE.

Propoſition VII. Problem.

The ſame things being ſuppoſed as in the laſt propoſition, it is required to find the center of the circle when the given point A is not in the line AC.

The conſtruction is immediately deduced from propoſition V. For if HI (Fig. 32.) be the line given in poſition, C the given point, and FDG the circle given in magnitude and poſition, it is obvious, that if on the ⊥ CA, LA be made = CL, and a circle be deſcribed to paſs through A, C, and cut the given circle as in F, G, the lines AC, FG being produced, and a tangent drawn to the given circle from their interſection B, ED being produced till it meets HI, will give the center of a circle M which will paſs through C and touch the circle GDF.

Propoſition VIII. Problem.

To find the center of a circle to paſs through two given points, and cut off from a circle given in magnitude and poſition a given arch.

Let

Let P, K be the two given points, (Fig. 32.) OGQ the given circle, and OQ the given arch. From N the center of the given circle defcribe an arch to touch OQ; and through P, K defcribe a circle cutting the given one in A, C; then produce AC, PK till they meet in B, from whence draw the tangent BF; and the points P, K, G, and F, will be in the periphery of the fame circle, and which will cut off from the given circle the arch GF = the affigned one.——The demonftration is the fame as in Propofition V.

Propofition IX. Problem. Fig. 34.

To find the center of a circle on a line given in pofition in refpect to a given circle, which may cut off from the given circle an affigned arch.

Let HI be the line given in pofition, and FGD the given circle. In any direction draw EP, and from T to F and G fet off half the given arch; then will a circle defcribed from P with diftance PF or PG evidently cut off from the given circle an arch equal to that affigned.——This propofition is unlimited; but is inferted on account of its frequent ufe in geometrical conftructions.

Propofition X. Problem.

The fame things being fuppofed as in the laft propofition, it is required to find the pofition of the center P when the circle is reftricted to a given magnitude.

From the former given circle cut off DL = to the affigned arch; and perpendicular to the chord DL draw the indefinite line EK, to which from the point D or L apply DK = the femidiameter of the other given circle P; then from E to the line HI apply EP = EK, and P will evidently be the pofition of the center as required.——A demonftration is needlefs.

From

From these few propositions may a great variety of Geometrical Problems be constructed, among which are the following ones in Burrow's Ladies' and Gentlemens' Diary for 1777 *.

Quest. 2.——The Data are reducible to this form,——The base, the vert. angle, and the rectangle of the sides.——For suppose AEC the required triangle (Fig. 35.) ED ⊥ to AC, and EB = EC; there are given AB, and ECA — EAC (= AEB); also $m : n :: EC\ (EB) : \frac{nEC}{m}$, and $AE \times \frac{nEB}{m} = MN$ a given rectangle; therefore $AE \times EB = \frac{m}{n} \times MN$, a given quantity.——Hence this construction,——On AB the given difference of the segments of the base, describe a segment of a circle containing an angle equal to the given difference of the angles at the base, and by Proposition III. find the point E so, that AE × EB may be equal to the given rectangle $\frac{m}{n} \times MN$; then on AB produced demit the perpend. ED, and make EC = EB; so will AEC be the required triangle.——Which is too evident to need any farther demonstration.

Limitation. $\frac{m}{n} \times MN$ must not exceed AB × BI.

Quest. 4.——From the end of the given base AB (Fig. 35.) draw BI equal thereto, making the angle ABI equal to the given one formed by the line drawn from the vertex to the base in D, and join A, I. From B with distance BE equal

* These propositions were drawn up in 1775, which was before the commencement of Burrow's Diary; and before I had seen the Rev. Mr. Lawson's *Dissertation on the Geometrical Analysis of the antients*, wherein I find we have both fallen upon the same method of constructing the 3d case of prop. II. being in effect the same with the 155 prop. of Pappus's 7th Book.——This I thought proper to mention, in order to prevent if possible, that too just charge against most modern writers——*plagiarism*.

the

the given difference of the ſides deſcribe the circle EF; and by Prop. VI. draw BC ſo, that CE may be = CA; and ACB will be the required triangle.——For from the propoſition BE = CB — AC which is equal the given difference by con. and becauſe the angle ABI is equal that given in the queſt. by con. and CD ‖ to BI (which is = BA) AD is evidently equal DC.

Remark. This queſtion ſeems to be improperly expreſſed, ſince from the other data the lines AD, DC admit of an equality; therefore they ſhould either have been reſtricted to *that*, or to a given ſum or difference.

Queſt. 10.——On AB the given intercepted line (Fig. 36.) deſcribe a circle ſuch, that the ſegment BAC may contain the given angle (Euc. 33. 3.); biſect AB in O, to which make OD perpendicular and equal the ſemidiameter of the given circle. Through D draw the indefinite line ML ‖ to AB; and by Prop. IV. through F draw EC = to the given diſtance in the queſtion. Then complete the figure by drawing CAL, CBM; and from E with diſtance DO deſcribing the circle PH, *and the thing is done.*——For AB is equal the given intercepted line, and ACB the given angle *per* con. and ſince AB is biſected by the ⊥ OD, the arches AF, BF are equal (Euc. 30. 3.) and therefore ACB is biſected by CE (Euc. 27. 3.) which is equal the given diſtance *per* conſtruction; and becauſe DO is equal the ſemidiameter of the given circle, and DM ‖ to AB, a circle deſcribed from center E and diſtance DO touches the line AB in P.——The min. limitation for AB in the data is obviouſly RS; but the max. is unlimited, for when AB becomes parallel to CL or CM it is infinite. Alſo from the conſtruction it appears that EC muſt not be greater than DQ; which are alſo both given.

After the ſame manner may one particular caſe of Queſt. 5. be conſtructed.———When BAC is a right angle; for then

NR

NR is given, being equal the diameter of the latter circle. Or it may be conſtructed thus.——From any point Q on the indefinite line NR (Fig. 37.) with the given diſtance QA deſcribe the circle NAR and make QP ⊥ to NR; on NR deſcribe the circular ſegment NFR as in Prop. I. to which apply NF = 2CD — NR. Then on AQ deſcribe the circle QT, and through P draw AD = the given diſtance, and produce indefinitely AR, AN; laſtly, from center D and diſtance equal the ſemidiameter of the given circle deſcribe the arch BC, and it will touch the lines AB, AC, and NR. Therefore a tangent NR is drawn to the given circle D, cutting the other given circle in Q, and NQ is = QR. The reaſon of which is obvious from the conſtruction.———It is obſervable that this queſtion as it ſtands in the Diary may be more elegantly expreſſed——The perimeter, the line biſecting the baſe, and the vert. angle being given, to conſtruct the triangle;—which has never yet been done, if we may rely on the generality of Lawſon's Synopſis.——It may however be effected in a manner almoſt ſimilar with the laſt conſtruction, and ſtrictly geometrical.—For making BAC = the given vert. angle, AB = AC = half the ſum of the ſides, and drawing the indefinite line AD to biſect the ang. BAD, if we biſect AB in S, draw HI ⊥ to AD, and take SK a fourth proportional to $\overline{AS—HI}$, $\overline{AH \backsim AS}$, and AS, then will QN be always a fourth proportional to KS, AS, and $\overline{KS—AQ \backsim AS}$. Conſequently QN is given, and therefore if on NR = 2QN we deſcribe the ſegment of a circle containing the given vert. angle, and from Q apply QA = the given line biſecting the baſe, or find the point F as in prop. I. (NF being now known) and join AN, AR, then will ANR be the required triangle. The demonſtration whereof may be eaſily deduced from the preceding propoſitions; and the limitations in the data are evident from the above analogies.

As

As this book, from its professed design, will naturally fall into the hands of those who are not much read in the mathematics, I suppose it will not be unacceptable to such to have a catalogue of those books that are generally esteemed the best on the subject, and which, it may reasonably be expected, with assiduity and perseverance, will soon make them proficients in the Mathematical Sciences.

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Astronomy. Dunn, Gregory, Heath*, Harris, Halley, Keill, Leadbetter, Long*, and Whiston. The Astronomical Tables of most note are, Halley's, 4to. 1752. 18s. Ferguson's 8vo. 2s. 6d. Mayer's 4to. 1770. 13s. sewed. Clairaut's 8vo. 1765. 3s. sewed. The Durham Tables, 4to 1766. 13s. 6d. boards.

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Book-keeping. Cooke, Everard, Gordon, Hutton, Mair, Perry, Quin, Roose*, and Webster*.

Chance, and Annuities. Buffon, Clark*, De Moivre*, Du Pre, Emerson, Halley, Price, and Simpson.

Chronology. Blair, Emerson, Ferguson, Du Fresnoy, Marshall, NEWTON, Strauchius, and Whiston.

Conics. De l'Hospital, De la Hire, De Wit, Emerson*, Hamilton, Jack*, Muller, Milne, Mydergius, Ozanam, Simpson, Vivani, Vincentio, Ward, and Wallis.

Decimals. Cunn, H. Clarke, Drape, Martin, Robertson, and Wilson.

Dialling. Emerson*, Ferguson, Leadbetter, Leybourn, Martin*, and Potter.

Electricity. Franklin, Freke, Ferguson, Hoadly, Lovett, Martin, Priestly*, Watson, Wesley*, and Wilson.

Fluxions. Ditton, Hayes, Maclaurin*, Muller, Sanderson, and Stone.

Fortification. Emerson, Muller*, Pleydell*, and Vauban.

Gauging. Clark*, Emerson, Leadbetter, Moss, Overley, Shirtcliffe, Symons*, and Turner.

Geography. Emerson, Fenning, Guthrie*, Jones*, Salmon, Varenius, and Wells.

Geometry. Anderson, Cunn, Donn*, D'Omerique, Fletcher, Gregory, Ghetaldus, Herigon, Keill, Lawson †, and Rudd.

† The Rev. Mr. Lawson, Rector of Swanscombe in Kent, who, among his other works, has published a most useful performance for the exercise of young Geometricians, intitled, A Dissertation on the Geometrical Analysis of the Antients. 1774. 2s. 6d. sewed.

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Mensuration. Fletcher Hawney, Hutton*, and Robertson.

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Music, the Mathematical Principles of, Emerson*, Helsham, and Martin.

Navigation. Atkinson, Crossby, Emerson, Harris, Martin, Maskelyne*, Patoun, Robertson*, Henry Wilson, and William Wilson*.

Optics. Baker, Emerson, Jurin, Martin, NEWTON, and Smith.

Perspective. H. Clarke, Emerson, s'Gravesande, Hamilton, Kirby, Lamy, Malton, Martin, Noble, Priestly, and Taylor. The *Latin* Authors are, Marolois, and Andrea Pozzo. The *Italian*, Daniello Barbaro, and Jacopo Barozzi da Vignola. The *French*, Cerceau, Jan Vredeman Friese, Lamy, and the Jesuit.

Philosophy. Bliss, Clare, Clarke, Cotes, Desaguliers, Green, s'Gravesande, Gregory, Hales, Helsham, Hawksbee, Fran. Jacquier, Keill, Muschenbroek, Abbé Nollet, Pemberton, Priestly, Rowning, Shaw, Stewart, Worster, Whiston, and Whiteside.

Surveying. Breaks*, Burn, Emerson, Fletcher, Gardiner*, Grey, Hammond, Lawrence, Wilde, and Wilson.

Trigonometry. Boad, Heys, Hawney, Payne, and Wilson. The Tables of most note are, Gardiner's, and Sherwin's. That Edition of Sherwin's, revised by Mr. Sam. Clark, is the most correct.

These

These Authors I have selected from among several others; but, as there are but few who have time and inclination to read so many, I have distinguished with an asterisk those which I dare venture to pronounce, from my own reading and occasional use of them, the best written treatises on the subject, and that will afford the most information in the least time.——Those Books, which are particularly distinguished by their Date, Price, &c. I would strenuously recommend a very careful and reiterated perusal thereof, as being undoubtedly, not only the best adapted to the capacity of a learner, but containing as it were the very basis on which the young mathematician is to build his whole superstructure.

ERRATA.

PAGE 32, *l.* 10, *r.* 999999; *p.* 95, *l.* 4, from bottom, for + in the num. *r.* $\times$; *p.* 127, *l.* 2, *r.* $-\frac{x^2}{8a}$, &c.; *p.* 167, *l.* 10, *r.* $\overline{\dot{F}E} = \dot{B}$; *p. ib.* *l.* 5, from bottom, *r.* *Ac*; *p.* 168, *l.* 4, *r.* C.

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EMENDATIONS.

The latter part of ART. 23, ſhould be as follows,—

—Add the terminate part to the numerator of the given fraction; remove the decimal point, and daſh off the repetends as before; then increaſe the right-hand place by the difference of the *new* terminate part and that which was added, and the circulate will be correct.

ART. 32, ſhould have been thus expreſſed,—

—If the circulates to be added have the places of their repetends in a geometrical progreſſion, of any ratio, the Sum, &c.